3.4. Application to differential equations. Let

$$\frac{d^n x}{dt^n}+a_1\frac{d^{n-1}x}{dt^{n-1}}+\ldots+a_{n-1}\frac{dx}{dt}+a_n x=f(t), \quad (1)$$

be a linear differential equation with constant coefficients, valid for $t>0$. Suppose we want a solution of this equation satisfying the conditions :

$$x=x_0,\ \frac{dx}{dt}=x_1,\ \frac{d^2x}{dt^2}=x_2,\ \ldots,\ \frac{d^{n-1}x}{dt^{n-1}}=x_{n-1}, \quad (2)$$

when $t=0$.

Multiply both sides of (1) by e^{-st} and integrate each term with respect to t from 0 to ∞. Then, using the result of § 3.3 and transposing the terms with minus sign to the right, we get

$$\begin{aligned}(s_n+a_1s^{n-1}+a_2s^{n-2}+&\ldots+a_{n-1}s+a_n)\bar{x}\\ =\bar{f}(s)&+(s^{n-1}+a_1s^{n-2}+\ldots+a_{n-2}s+a_{n-1})x_0\\ &+(s^{n-2}+a_1s^{n-3}+\ldots+a_{n-2})x_1\\ &+(s^{n-3}+a_1s^{n-4}+\ldots+a_{n-3})x_2\\ &+\ldots+(s+a_1)x_{n-2}+x_{n-1}, \quad (3)\end{aligned}$$

where $\bar{x}$ denotes the Laplace transform of x and $\bar{f}(s)$ that of $f(t)$.

Dividing by $(s^n+a_1s^{n-1}+\ldots+a_n)$, we get $\bar{x}$ as a known function of s. Resolving this into partial fractions and taking its inverse, we get x as a function of t. This will be a solution of (1) satisfying the initial conditions (2).

Equation (3) is called the 'subsidiary equation'. If the given differential equation is written as

$$(D^n+a_1D^{n-1}+\ldots+a_{n-1}D+a_n)x=f(t),$$

the left-hand side of the subsidiary equation is obtained by replacing D by s and x by $\bar{x}$. The first term on the right is $\bar{f}(s)$, the transform of $f(t)$. The remaining terms on the right-hand side are terms in x_0, x_1, x_2, ... x_{n-1}

multiplied by polyomials in s. These polynomials are obtained by dividing

$$s^n+a_1s^{n-1}+\ \dots\ +a_{n-1}s+a_n$$

(i.e. the factor multiplying $\bar{x}$ on the left) successively by $s, s^2, s^3, \dots, s^n$, and dropping off any negative powers of s.

Ex. 1. Solve the equation

$$(D^2-2D+2)x=0,\ x_0=x_1=1. \qquad \text{[\textit{Banaras}, 1971]}$$

On Laplace transformation the given equation becomes

$$(s^2-2s+2)\bar{x} = 0+(s-2)x_0+x_1$$
$$= (s-2)+1=s-1.$$

Therefore $$\bar{x}=\frac{s-1}{s^2-2s+2}=\frac{s-1}{(s-1)^2+1}.$$

On inversion, we get

$$x=e^t \cos t.$$

Ex. 2. Solve the equation

$$(D-1)(D-2)(D-3)x=5,\ x_0=0,\ x_1=1,\ x_2=0.$$

The left-hand side is $(D^3-6D^2+11D-6)x$; so transformation gives

$$(s-1)(s-2)(s-3)\bar{x} = \frac{5}{s}+(s^2-6s+11)x_0+(s-6)x_1+x_2$$
$$= \frac{5}{s}+(s-6)=\frac{s^2-6s+5}{s}.$$

Therefore $$\bar{x} = \frac{s^2-6s+5}{s(s-1)(s-2)(s-3)}$$
$$= \frac{5}{s(-6)}+\frac{0}{s-1}+\frac{(-3)}{(-2)(s-2)}+\frac{(-4)}{6(s-3)},$$

on breaking into partial fractions. Inversion gives

$$x=\tfrac{3}{2}e^{2t}-\tfrac{2}{3}e^{3t}-\tfrac{5}{6}.$$

Examples 2.6

Q.No.1 The given equation is

$$(D^2+3D+2)x=0, \ x=x_0, \ Dx=x_1 \ \text{at } t=0 \quad —(1)$$

The laplace transform of (1) gives

$$(s^2+3s+2)\bar{x} = (s+3)x_0+x_1$$

$$\therefore \quad \bar{x} = \frac{(s+3)x_0+x_1}{(s^2+3s+2)} = \frac{(s+3)x_0+x_1}{(s+1)(s+2)}$$

$$\bar{x} = \frac{2x_0+x_1}{s+1} - \frac{x_0+x_1}{s+2}$$ by partial fraction

$$\therefore L^{-1}(\bar{x}) = L^{-1}\left\{\frac{2x_0+x_1}{s+1}\right\} - L^{-1}\left\{\frac{x_0+x_1}{s+2}\right\}$$

$$x = (2x_0+x_1)e^{-t} - (x_0+x_1)e^{-2t}$$

~~This~~ which is the required solution of (1)

Q.No.2 $(D^2-3D+2)x = e^{at}, \ a\neq 1, 2, \ x=x_0, \ Dx=x_1$ at $t=0$ —(1)

The laplace transforms of equation (1) gives

$$(s^2-3s+2)\bar{x} = \frac{1}{s-a} + (s-3)x_0+x_1$$

$$\therefore \quad \bar{x} = \frac{1}{(s-a)(s^2-3s+2)} + \frac{(s-3)x_0+x_1}{(s^2-3s+2)}$$

$$= \frac{1}{(s-a)(s-1)(s-2)} + \frac{(s-3)x_0+x_1}{(s-1)(s-2)}$$

$-2x_0+x_1$ / -1 / x_0-x_1

$$\bar{x} = \frac{1}{(s-a)(a-1)(a-2)} + \frac{1}{(a-1)(s-1)} - \frac{1}{(a-2)(s-1)} + \frac{2x_0+x_1}{s-1} - \frac{x_0+x_1}{s-2}$$

$$L^{-1}\{\bar{x}\} = L^{-1}\left\{\frac{1}{(s-a)(a-1)(a-2)} + \frac{1}{(a-1)(s-1)} - \frac{1}{(a-2)(s-2)} + \frac{2x_0+x_1}{s-1} - \frac{x_0+x_1}{s-2}\right\}$$

$$x = \frac{1}{(a-1)(a-2)}e^{at} + \frac{1}{(a-1)}e^{t} - \frac{1}{(a-1)}e^{2t}$$

$$+(2x_0+x_1)e^{t} - (x_0+x_1)e^{2t}$$

$$x = \frac{1}{(a-1)(a-2)}e^{at} + \left(2x_0+x_1+\frac{1}{(a-1)}\right)e^{t}$$

$$-\left(x_0+x_1+\frac{1}{a-2}\right)e^{2t}$$

which is the required solution of (1)

Q.NO.3 $(D^2-3D+2)x = 1-e^{2t}$, $x_0=1, x_1=0$ —— (1)

The laplace transform of (1) gives

$$(s^2-3s+2)\bar{x} = \frac{1}{s} - \frac{1}{s-2} + (s-3)x_0 + x_1$$

$$= \frac{1}{s} - \frac{1}{s-2} + (s-3)$$

$$= \frac{s-2-s+s(s-2)(s-3)}{s(s-2)}$$

$$= \frac{s^3-5s^2+6s-2}{s(s-2)}$$

$$\therefore \quad \bar{x} = \frac{s^3-5s^2+6s-2}{s(s-1)(s-2)^2}$$

$$= \frac{A}{s} + \frac{B}{s-1} + \frac{C}{(s-2)^2} + \frac{D}{(s-2)}$$

$B=0$, $C=-1$, $A=\frac{1}{2}$,

Multiplying both sides by s and letting $s \to \infty$ we have

$$1 = \frac{1}{2}+D \text{ or. } D = 1/2$$

$$\therefore \bar{x} = \frac{1}{2s} + \frac{0}{s-1} - \frac{1}{(s-2)^2} + \frac{1}{2(s-2)}$$

$$\therefore L^{-1}\{\bar{x}\} = \frac{1}{2} - t\,e^{2t} + \frac{1}{2}e^{2t}$$

$$x = \frac{1}{2} + \left(-t+\frac{1}{2}\right).e^{2t}$$

This is the required solution.

Q.No.10 $(D^2-D-2)x = 20\sin 2t$, $x_0=-1$, $x_1=2$ — (1)

The laplace transforms of (1) gives

$$(s^2-s-2)\bar{x} = \frac{20\times 2}{s^2+4} + (s-1)x_0 + x_1$$

$$= \frac{40}{s^2+4} + (s-1)(-1) + 2$$

$$= \frac{40}{s^2+4} - (s-1) + 2 = \frac{40}{s^2+4} - (s-3)$$

$$\cancel{\therefore \bar{x} = \frac{40}{(s^2+4)(s^2-s-2)} + \frac{(s-1)}{(s^2-s-2)} + \frac{2}{(s^2-s-2)}}$$

$$= \frac{\cancel{40-(s^3+4s-s^2-4)+2s^2+8}}{s^2+4}$$

$$= \frac{\cancel{40-s^3+3s^2-4s+12}}{s^2+4} = \frac{40+(s^2+4)(s-3)}{s^2+4}$$

$$\therefore \bar{x} = \frac{40+(s^2+4)(s-3)}{(s^2+4)(s^2-s-2)} = \frac{40+(s^2+4)(s-3)}{(s+1)(s-2)(s^2+4)}$$

$$= \frac{A}{s+1} + \frac{B}{s-2} + \frac{Cs+D}{s^2+4}$$

$A = 60/-15 = -4$, $B = 2$

Put. ~~s=0 and s=0, we get~~ To determine C and D multiplying by s^2+4 on both sides and put $s=2i$

$\cancel{-1 = -4+2+C \text{ and } -13/2 = -4-1+\frac{1}{4}D}$

we get $C=1$ and $D=-6$

Hence

$$\bar{x} = -\frac{4}{s+1} + \frac{2}{(s-2)} + \frac{s-6}{(s^2+4)}$$

Taking inverse laplace transform we get,

$$L^{-1}\{\bar{x}\} = -4L^{-1}\left\{\frac{1}{s+1}\right\} + 2L^{-1}\left\{\frac{1}{s-2}\right\} + \left[L^{-1}\left\{\frac{s}{s^2+4}\right\} - 6L^{-1}\left\{\frac{1}{s^2+4}\right\}\right]$$

$$x = -4e^{-t} + 2e^{2t} + \left[\cos 2t - \frac{6}{2}\sin 2t\right]$$

$$x = -4e^{-t} + 2e^{2t} + \left[\cos 2t - 3\sin 2t\right]$$

is the required solution of equation (1)

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Q.NO 13. $(D^3 - D)x = 2\cos t \quad , x_0 = 3, x_1 = 2, x_2 = 1$ — ①

The Laplace transforms of the equation ① gives

$$(s^3 - s)\bar{x} = \frac{2s}{s^2+1} + (s^2-1)x_0 + sx_1 + x_2$$

$$= \frac{2s}{s^2+1} + 3(s^2-1) + 2s + 1$$

$$\therefore \bar{x} = \frac{2s}{(s^2+1)(s^3-s)} + \frac{3}{s} + \frac{2s}{s(s^2-1)} + \frac{1}{s(s^2-1)}$$

$$\bar{x} = \frac{2}{(s^2+1)(s^2-1)} + \frac{3}{s} + \frac{2}{s^2-1} + \frac{1}{s(s^2-1)}$$

$$= \left(\frac{1}{s^2-1} - \frac{1}{s^2+1}\right) + \frac{3}{s} + \frac{2}{s^2-1} + \left(\frac{s}{s^2-1} - \frac{1}{s}\right)$$

(By inspection)

Taking inverse Laplace transforms we get

$$L^{-1}\{\bar{x}\} = L^{-1}\left\{\frac{1}{s^2-1}\right\} - L^{-1}\left\{\frac{1}{s^2+1}\right\} + 3L^{-1}\left\{\frac{1}{s}\right\} + 2L^{-1}\left\{\frac{1}{s^2-1}\right\} + L^{-1}\left\{\frac{s}{s^2-1}\right\} - L^{-1}\left\{\frac{1}{s}\right\}$$

$$x = \sinh t - \sin t + 3 + 2\sinh t + \cosh t - 1$$

$$x = 3\sinh t + \cosh t - \sin t + 2$$

which is the required solution of the given equation ①

Q.No. 14 $(D^3 - D^2 + 4D - 4)x = 68e^t \sin 2t, \ x_0 = 1, x_1 = -19, x_2 = -37$ — ①

The Laplace transform of ① gives

$$(s^3 - s^2 + 4s - 4)\bar{x} = 68\frac{2}{(s-1)^2+2^2} + (s^2-s+4)x_0 + (s-1)x_1 + x_2$$

$$\left(\text{Since } L\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}\right)$$

$$= \frac{136}{(s-1)^2+4} + (s^2-s+4)(+1) + (s-1)(-19) - 37$$

$$= \frac{136}{(s-1)^2+4} + s^2 - s + 4 - 19s + 19 - 37 = \frac{136}{(s-1)^2+4} + s^2 - 20s - 14$$

$$= \frac{136 + (s^2-2s+5)(s^2-20s-14)}{(s^2-2s+5)}$$

$$\therefore \bar{x} = \frac{136 + (s^2-2s+5)(s^2-20s-14)}{(s^2-2s+5)(s^3-s^2+4s-4)}$$

$$= \frac{136+(s^2-2s+5)(s^2-20s-14)}{(s-1)(s^2+4)(s^2-2s+5)}$$

$$= \frac{A}{s-1} + \frac{Bs+C}{s^2+4} + \frac{Ds+E}{(s^2-2s+5)}$$

$A = 1/5$, and for obtaining the constants B and C, multiplying both sides by s^2+4 and put $s^2=-4$, then we get

$$\frac{136+(-2s+1)(-20s-18)}{(s-1)(-2s+1)} = Bs+C$$

$$136+40s^2+36s-20s-18 = (Bs+C)(-2s^2+3s-1)$$

$$136-160+6s-18 = (Bs+C)(3s+7)$$

$$6s-42 = 3Bs^2+7Bs+3Cs+7C$$

$$6s-42 = -12B+\underline{7Bs+3Cs}+7C$$

$7B+3C = 6$ —② and $-12B+7C = -42$ —③

solving ② and ③ we get

$B = 14/5$, $C = -6/5$

Now for obtaining the constant D, multiplying ① by s and let $s \to \infty$, then we get
$D = -2$

Also put $s=0$ in ①, then we get
$E = -14$

Hence,

$$\bar{x} = \frac{1}{5(s-1)} + \frac{14s-6}{5(s^2+4)} + \frac{-2s-14}{s^2-2s+5}$$

$$\text{or } \bar{x} = \frac{1}{5}\left[\frac{1}{s-1} + \frac{14s}{s^2+4} - \frac{6}{s^2+4}\right] - 2\left\{\frac{s-1}{(s-1)^2+4} + \frac{8}{(s-1)^2+4}\right\}$$ —④

Now taking the inverse Laplace transforms of ④ we get —

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$$L^{-1}\{\bar{x}\} = \frac{1}{5}\left[L^{-1}\left\{\frac{1}{s-1}\right\} + 14\,L^{-1}\left\{\frac{s}{s^2+4}\right\} - 6\,L^{-1}\left\{\frac{2}{2(s^2+4)}\right\}\right]$$

$$-2\left[L^{-1}\left\{\frac{s-1}{(s-1)^2+4}\right\} + 4\,L^{-1}\left\{\frac{2}{(s-1)^2+4}\right\}\right]$$

$$x = \frac{1}{5}\left[e^t + 14\cos 2t - 3\sin 2t\right] - 2\left[e^t\cos 2t + 4e^t\sin 2t\right]$$

$$x = \frac{1}{5}\left[e^t + 14\cos 2t - 3\sin 2t\right] - 2e^t\left[\cos 2t + 4\sin 2t\right]$$

which is the required solution of the given equation ①.

Q.No.16 $(D^4-1)x = 1, \quad x_0 = x_1 = x_2 = x_3 = 0$ — ①

The laplace transform of ① gives

$$(s^4-1)\bar{x} = \frac{1}{s} + s^3x_0 + s^2x_1 + sx_2 + x_3$$

$$= \frac{1}{s}$$

$$\therefore \bar{x} = \frac{1}{s(s^4-1)} = \frac{1}{s(s^2-1)(s^2+1)} = \frac{s}{s^2(s^2-1)(s^2+1)}$$

$$= s\left[\frac{1}{2(s^2-1)} - \frac{1}{s^2} + \frac{1}{2(s^2+1)}\right]$$

on breaking into partial fractions linear in s^2.

$$\bar{x} = \frac{s}{2(s^2-1)} - \frac{1}{s} + \frac{s}{2(s^2+1)}.$$

Now taking inverse laplace transforms we get.

$$L^{-1}\{\bar{x}\} = \frac{1}{2}L^{-1}\left\{\frac{s}{s^2-1}\right\} - L^{-1}\left\{\frac{1}{s}\right\} + \frac{1}{2}L^{-1}\left\{\frac{s}{s^2+1}\right\}$$

$$x = \frac{1}{2}\cosh t - 1 + \frac{1}{2}\cos t = \frac{1}{2}(\cosh t + \cos t) - 1$$

is the required solution of ①.

Q.No.17 $(D^3-2D^2+5D)x = 0, \quad x=0,\ Dx=1,\ \text{at } t=0,\ x=1,\ \text{at } t=\pi/8.$ — ①

The laplace transforms of ① gives

$$(s^3-2s^2+5s)\bar{x} = 0 + (s^2-2s+5)x_0 + (s-2)x_1 + x_2$$

$$(s^3-2s^2+5s)\bar{x} = (s^2-2s+5)\times 0 + (s-2)\times 1 + x_2$$

$$\therefore \quad \bar{x} = \frac{s-2+x_2}{s(s^2-2s+5)}$$

$$= \frac{A}{s} + \frac{Bs+C}{s^2-2s+5}$$

$$A = \frac{x_2-2}{5},$$

Multiplying both sides by s and let $s \to \infty$, then

$$\bar{x} = \frac{x_2-2}{5s} + \frac{Bs+C}{s^2-2s+5} \quad \text{—— (1)}$$

$$B = -\frac{x_2-2}{5}.$$

$$0 = \frac{x_2-2}{5} + B \Rightarrow B = -\frac{x_2-2}{5}$$

$$\text{or} \quad B = \frac{2-x_2}{5} \quad \text{—— (2)}$$

Put $s=1$ in (1), we have

$$\frac{x_2-1}{4} = \frac{x_2-2}{5} + \frac{(2-x_2)+5C}{20}$$

$$\cancel{= A + \frac{-5A+C}{4}}$$

$$\cancel{(x_2-1) = 4A - 5A + C}$$

$$\cancel{C = (x_2-1)+A = (x_2-1) + \frac{x_2-2}{5}}$$

$$\cancel{= \frac{5x_2-5+x_2-2}{5} = \frac{6x_2-7}{5}}$$

$$5(x_2-1) = 4(x_2-2) + (2-x_2) + 5C$$

$$5x_2-5-4x_2+8-2+x_2 = 5C$$

$$2x_2+1 = 5C \Rightarrow C = (2x_2+1)/5$$

Hence

$$\bar{x} = \frac{x_2-2}{5s} + \frac{\frac{1}{5}(2-x_2)s + \frac{1}{5}(2x_2+1)}{(s^2-2s+5)}$$

$$= \left(\frac{x_2-2}{5}\right)\cdot\frac{1}{s} + \frac{(2-x_2)s+(2x_2+1)}{5(s^2-2s+5)}$$

$$\bar{x} = \left(\frac{x_2-2}{5}\right)\frac{1}{s} + \frac{(2-x_2)(s-1)}{5\{(s-1)^2+4\}} + \frac{x_2+3}{5\{(s-1)^2+4\}}$$

$$\therefore L^{-1}\{\bar{x}\} = \left(\frac{x_2-2}{5}\right)L^{-1}\left\{\frac{1}{s}\right\} + \left(\frac{2-x_2}{5}\right)L^{-1}\left\{\frac{s-1}{(s-1)^2+4}\right\} + \left(\frac{x_2+3}{10}\right)L^{-1}\left\{\frac{2}{(s-1)^2+4}\right\}$$

$$x = \left(\frac{x_2-2}{5}\right)\times 1 + \left(\frac{2-x_2}{5}\right)e^t\cos 2t + \left(\frac{x_2+3}{10}\right)e^t\sin 2t$$

or $\bar{x} = \frac{1}{5}\left[(x_2-2)+(2-x_2)\, e^{t}\cos 2t + \frac{1}{2}(x_2+3)\, e^{t}\sin 2t \right]$ --- (4)

Also $x=1$ at $t=\pi/8$

Put $x=1$, $t=\pi/8$ in (4) we get.

$$1 = \frac{1}{5}\left[(x_2-2)+(2-x_2)\, e^{\pi/8}\cos \pi/4 + \frac{1}{2}(x_2+3)\, e^{\pi/8}\sin \pi/4 \right]$$

$$1 = \frac{1}{5}\left[(x_2-2)+(2-x_2)\, e^{\pi/8}\cdot\frac{1}{\sqrt{2}} + \frac{1}{2}(x_2+3)\, e^{\pi/8}\cdot\frac{1}{\sqrt{2}} \right]$$

$$= \frac{1}{5}\left[(x_2-2) + \frac{1}{2}(4-2x_2+x_2+3)\, e^{\pi/8}\cdot\frac{1}{\sqrt{2}} \right]$$

$$1 = \frac{1}{5}\left[(x_2-2) + \frac{1}{2}(7-x_2)\, e^{\pi/8}\cdot\frac{1}{\sqrt{2}} \right]$$

$$10 = 2x_2 - 4 + (7-x_2)\, e^{\pi/8}\cdot\frac{1}{\sqrt{2}}$$

$$14 = 2x_2 + 7e^{\pi/8}\cdot\frac{1}{\sqrt{2}} - e^{\pi/8}\cdot\frac{1}{\sqrt{2}}x_2$$

$$= \left(2 - e^{\pi/8}\cdot\frac{1}{\sqrt{2}}\right)x_2 + 7e^{\pi/8}\frac{1}{\sqrt{2}}$$

$$14 - 7e^{\pi/8}\cdot\frac{1}{\sqrt{2}} =$$

$$7\cancel{\left(2 - e^{\pi/8}\frac{1}{\sqrt{2}}\right)} = \cancel{\left(2 - e^{\pi/8}\cdot\frac{1}{\sqrt{2}}\right)}x_2$$

$$\therefore\ x_2 = 7.$$

Now put the value of x_2 in (4) we get. the final solution of (1).

$$x = \frac{1}{5}\left[5 + (2-7)\, e^{t}\cos 2t + \frac{1}{2}(7+3)\, e^{t}\sin 2t \right]$$

$$= \frac{1}{5}\left[5 - 5e^{t}\cos 2t + 5\, e^{t}\sin 2t \right]$$

$$x = \left(1 - e^{t}\cos 2t + e^{t}\sin 2t\right)$$

Q.No.18 $d^2y/dx^2 + 9y = \cos 2t$, $y(0)=1$, $y(\pi/2)=-1$.

or $(D^2+9)\,y = \cos 2t$, $y_0=1$ --- (1)

The Laplace transforms of (1) gives

$$(s^2+9)\bar{y} = \frac{s}{s^2+2^2} + s y_0 + y_1$$

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$$(s^2+9)\bar{y} = \frac{s}{s^2+4} + s.1 + y_1$$

$$= \frac{s}{s^2+4} + s + y_1.$$

or $$\bar{y} = \frac{s}{(s^2+4)(s^2+9)} + \frac{s}{s^2+9} + \frac{y_1}{s^2+9}$$

$$= \frac{1}{5}\left(\frac{s}{s^2+4} - \frac{s}{s^2+9}\right) + \frac{s}{s^2+9} + \frac{y_1}{s^2+9}$$

(by inspection)

Taking the inverse laplace transforms we get

$$L^{-1}\{\bar{y}\} = \frac{1}{5}\left[L^{-1}\left\{\frac{s}{s^2+4} - \frac{s}{s^2+9}\right\}\right] + L^{-1}\left\{\frac{s}{s^2+9}\right\} + \frac{y_1}{3} L^{-1}\left\{\frac{3}{s^2+9}\right\}$$

$$y = \frac{1}{5}\left[\cos 2t - \cos 3t\right] + \cos 3t + \frac{1}{3} y_1 \sin 3t$$

$$y = \frac{1}{5}\left(\cos 2t + 4\cos 3t\right) + \frac{1}{3} y_1 \sin 3t \quad \text{—— (2)}$$

But $y = -1$ at $t = \pi/2$, so put $y = -1$ and $t = \pi/2$ in (2) we get

$$-1 = \frac{1}{5}\left(\cos\pi + 4\cos 3\pi/2\right) + \frac{1}{3} y_1 \sin 3\pi/2$$

$$-1 = \frac{1}{5}\left(-1 + 4\times 0\right) + \frac{1}{3} y_1 (-1)$$

$$+1 = -1/5 - 1/3\, y_1 = +\left(\frac{1}{5} + \frac{1}{3} y_1\right)$$

$$15 = 3 + 5y_1 \Rightarrow y_1 = 12/5$$

Put the value of y_1 in (2) we get the final solution of the given equation (1)

$$y = \frac{1}{5}\left(\cos 2t + 4\cos 3t\right) + \frac{1}{3}\cdot\frac{12}{5}\sin 3t$$

$$y = \frac{1}{5}\left(\cos 2t + 4\cos 3t\right) + 4/5 \sin 3t$$

Q.No. 19 $y'' + 2y' + 5y = e^{-t}\sin t$, $y(0) = 0$, $y'(0) = 1$

or $(D^2+2D+5)y = e^{-t}\sin t$, $y(0)=0$, $y'(0)=1$ — (1)

The Laplace transforms of (1) gives

$$(s^2+2s+5)\bar{y} = \frac{1}{(s+1)^2+1} + (s+2)y_0 + 1$$

$$= \frac{1}{s^2+2s+2} + 1$$

$$\therefore \bar{y} = \frac{1}{(s^2+2s+2)(s^2+2s+5)} + \frac{1}{(s^2+2s+5)}$$

$$= \frac{1}{3}\left(\frac{1}{s^2+2s+2} - \frac{1}{s^2+2s+5}\right) + \frac{2}{2[(s+1)^2+2^2]}$$

$$\bar{y} = \frac{1}{3}\left[\frac{1}{(s+1)^2+1} - \frac{2}{2(s+1)^2+2^2}\right] + \frac{1}{2}\,\frac{2}{(s+1)^2+2^2} \quad \text{— (2)}$$

Taking the inverse Laplace transforms of (2) we get,

$$y = \frac{1}{3}\left[e^{-t}\sin t - \frac{1}{2}e^{-t}\sin 2t\right] + \frac{1}{2}e^{-t}\sin 2t$$

$$= \frac{1}{3}e^{-t}\sin t - \frac{1}{6}e^{-t}\sin 2t + \frac{1}{2}e^{-t}\sin 2t$$

$$= \frac{1}{3}e^{-t}\sin t - \frac{1}{6}\left(e^{-t}\sin 2t - 3e^{-t}\sin 2t\right)$$

$$= \frac{1}{3}e^{-t}\sin t - \frac{1}{6}\left(-2e^{-t}\sin 2t\right)$$

$y = \frac{1}{3}e^{-t}(\sin t + \sin 2t)$ is the required solution of (1)

Q.No. 20 $y''' + 2y'' - y' - 2y = 0$, $y(0) = y'(0) = 0$, $y''(0) = 6$

or $(D^3+2D^2-D-2)y = 0$, $y_0 = 0$, $y_1 = 0$, $y_2 = 6$ — (1)

Laplace transforms of (1) gives.

$$(s^3+2s^2-s-2)\bar{y} = 0 + (s^2+2s-1)y_0 + (s+2)y_1 + y_2$$

$$= 0 + 0 + 0 + 6$$

$$\bar{y} = \frac{6}{s^3+2s^2-s-2} = \frac{6}{(s^2-1)(s+2)} = \frac{6}{(s-1)(s+1)(s+2)}$$

$$= \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s+2}$$ by partial fraction.