

PARTIAL DIFFERENTIATION.

Let $f(x, y)$ be a function of two variables x and y
 $z = f(x, y)$.

$$\frac{\partial z}{\partial x}$$

Differentiation $\left\{ \begin{array}{l} \text{Ordinary} \\ \text{Partial (dependent on more than 1 variable).} \end{array} \right.$

• $y = f(x)$

by first principle-

$$\frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

$$\frac{\partial z}{\partial x} = \lim_{\delta x \rightarrow 0} \frac{f(x + \delta x, y) - f(x, y)}{\delta x} \quad y \rightarrow \text{constant}$$

$$\frac{\partial z}{\partial y} = \lim_{\delta y \rightarrow 0} \frac{f(x, y + \delta y) - f(x, y)}{\delta y} \quad x \rightarrow \text{const.}$$

• $z = f(x_1, x_2, \dots, x_n)$ $x_1, x_2, \dots, x_n \rightarrow \text{independent}$
 z depends on $x_1, x_2, \dots, x_n$.

$$\frac{\partial z}{\partial x_1} = \lim_{\delta x_1 \rightarrow 0} \frac{f(x_1 + \delta x_1, x_2, x_3, \dots, x_n) - f(x_1, x_2, \dots, x_n)}{\delta x_1}$$

ex-

$$u = e^{xyz}$$

$$\frac{\partial u}{\partial x} = yz \cdot e^{xyz}$$

$$\frac{\partial u}{\partial y} = xz \cdot e^{xyz}$$

$$\frac{\partial u}{\partial z} = xy \cdot e^{xyz}$$

• $f_x, f_y, f_{xy}, f_{yx}, f_{xx}, f_{yy}$
 $\rightarrow$ Partial diff.

• $f_{xy} = f_{yx}$ when
 fnd. is continuous.

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right)$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right)$$

Homogeneous function - (All terms have same degree)

$$f(x, y) = a_0 x^2 + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \dots + a_n \left(\frac{y}{x}\right)^n$$

$$f(x, y) = x^n \left[a_0 + a_1 \left(\frac{y}{x}\right) + a_2 \left(\frac{y}{x}\right)^2 + \dots + a_n \left(\frac{y}{x}\right)^n \right]$$

$$= x^n \phi\left(\frac{y}{x}\right)$$

f is said to be homogeneous function whenever f can be written as $x^n \phi\left(\frac{y}{x}\right)$ of degree n.

$$f(x, y) = \frac{x^2 + 2xy + y^2}{x^2} = x^2 \left[1 + 2\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2 \right]$$

$$= x^2 \phi\left(\frac{y}{x}\right)$$

$$f(x, y) = \phi\left(\frac{y}{x}\right) \text{ where } \phi\left(\frac{y}{x}\right) \neq 0 \forall x, y$$

• Rational function is hom. when both nr. & dr. is homo. & degree = Nr. - dr.

$$f(x, y) = \frac{x^3 + 2x^2 y + y^3}{x^2 + 2x^2 y + y^2}$$

$$= x^3 \left[1 + 2\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2 \right]$$

$$= x^3 \left[1 + 2\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2 \right]$$

$$= x^3 + 2x^2 y + x y^2$$

$$\text{degree} = -1$$

$$\frac{\partial f}{\partial x} = \lim_{sx \rightarrow 0} \frac{f(x+sx, y) - f(x, y)}{sx}$$

$$\frac{\partial f}{\partial y} = \lim_{sy \rightarrow 0} \frac{f(x, y+sy) - f(x, y)}{sy}$$

$$\text{Ex. } f(x, y) = x^2 + y^2 + x$$

$$\frac{\partial f}{\partial x} = \lim_{sx \rightarrow 0} \frac{f(x+sx, y) - f(x, y)}{sx}$$

$$= \lim_{sx \rightarrow 0} \frac{(x+sx)^2 + y^2 + (x+sx) - (x^2 + y^2 + x)}{sx}$$

$$= \lim_{sx \rightarrow 0} \frac{(2x+1)sx + sx^2}{sx}$$

$$= \lim_{sx \rightarrow 0} [(2x+1) + sx]$$

$$(7x+1)$$

$$= 2x+1$$

Ex-2.3/3. For the function $f(x, y) = \begin{cases} y(x^2 - y^2) & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$

$$\frac{\partial f}{\partial x} = 2 \left(\frac{\partial f}{\partial x} \right)$$

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Homogeneous function - All terms have same degree

$$f(x, y) = 0 \cdot x^2 + 0 \cdot x^2 y + 0 \cdot x^2 y^2 + \dots + 0 \cdot y^2$$

$$f(x, y) = x^2 \left(0 + 0 \cdot \frac{y}{x} + 0 \cdot \left(\frac{y}{x} \right)^2 + \dots + 0 \cdot \left(\frac{y}{x} \right)^2 \right)$$

$$= x^2 \left(\frac{y^2}{x^2} \right)$$

f is said to be homogeneous function when f(x, y) is of degree n

$$f(x, y) = \frac{x^2 + 3xy + y^2}{x^2} = \left(\frac{y}{x} \right)^2$$

$$= y^2 \left(\frac{1}{x} \right)^2$$

$$f(x, y) = f\left(\frac{x}{\lambda}, \frac{y}{\lambda}\right) \text{ where } \lambda(x, y) = 0 \cdot x + y$$

$$(211)$$

$$f(x, y) = \frac{x^2 + 3xy + y^2}{x^2} \text{ for the function } f(x, y) = \begin{cases} \frac{x^2 + 3xy + y^2}{x^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

$$f(x, y) = \frac{x^2 + 3xy + y^2}{x^2}$$

$$= \frac{x^2}{x^2} + \frac{3xy}{x^2} + \frac{y^2}{x^2}$$

$$= 1 + 3 \left(\frac{y}{x} \right) + \left(\frac{y}{x} \right)^2$$

$$= 1 + 3 \left(\frac{y}{x} \right) + \left(\frac{y}{x} \right)^2$$

$$\text{Degree} = 1$$

$$\frac{\partial f}{\partial x} = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$\frac{\partial f}{\partial y} = \lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h}$$

$$\frac{\partial}{\partial x} \left(\frac{x^2 + 3xy + y^2}{x^2} \right) = \frac{x^2 + 3xy + y^2}{x^2}$$

$$= \lim_{h \rightarrow 0} \frac{(x+h)^2 + 3(x+h)y + y^2}{(x+h)^2} - \frac{(x^2 + 3xy + y^2)}{x^2}$$

$$= \lim_{h \rightarrow 0} \frac{(x^2 + 2xh + h^2) + 3xy + 3hy + y^2}{x^2 + 2xh + h^2} - \frac{(x^2 + 3xy + y^2)}{x^2}$$

$$= \frac{1}{x^2} [3x^2 + 3y^2]$$

$$= \frac{3y^2}{x^2}$$

Compute $f_x(a,0)$, $f_y(a,0)$, $f_z(a,0)$, $f_{xx}(a,0)$ if #

$$f(x,y) = \begin{cases} \frac{y^2(x^2+y^2)}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$$f_x = \frac{\partial f}{\partial x} = \lim_{s_1 \rightarrow 0} \frac{f(x+s_1, y) - f(x, y)}{s_1}$$

$$= \lim_{s_1 \rightarrow 0} \frac{y^2 \left[\frac{(x+s_1)^2 + y^2}{(x+s_1)^2 + y^2} - \frac{x^2 + y^2}{x^2 + y^2} \right]}{s_1}$$

$$f_x(x,y) = \lim_{s_1 \rightarrow 0} \frac{y^2 \left[\frac{(x+s_1)^2 + y^2}{(x+s_1)^2 + y^2} - \frac{x^2 + y^2}{x^2 + y^2} \right]}{s_1} + y$$

$$= \lim_{s_1 \rightarrow 0} \frac{y^2 (s_1^2 - y^2)}{(s_1^2 + y^2) s_1} + \frac{y}{s_1}$$

$$= \lim_{s_1 \rightarrow 0} \frac{y^2 s_1}{s_1^2 + y^2} + \frac{y}{s_1}$$

not define

$$f_x = \lim_{s_1 \rightarrow 0} \frac{f(x+s_1, y) - f(x, y)}{s_1}$$

$$f_x(a,0) = \lim_{s_1 \rightarrow 0} \frac{f(s_1, 0) - f(0,0)}{s_1}$$

$$\lim_{s_1 \rightarrow 0} \frac{0 - 0}{s_1}$$

$$f_y = \lim_{s_2 \rightarrow 0} \frac{f(x, y+s_2) - f(x, y)}{s_2}$$

$$= \lim_{s_2 \rightarrow 0} \frac{(y+s_2)^2 \frac{x^2 + (y+s_2)^2}{x^2 + (y+s_2)^2} - y^2 \frac{x^2 + y^2}{x^2 + y^2}}{s_2}$$

$$f_y(x,0) = \lim_{s_2 \rightarrow 0} \frac{(0+s_2)^2 \frac{x^2 + s_2^2}{x^2 + s_2^2} - 0}{s_2}$$

$$= \lim_{s_2 \rightarrow 0} \frac{x^2 - s_2^2}{x^2 + s_2^2}$$

$$= \lim_{s_2 \rightarrow 0} 1$$

$$f_y(a,0) = \lim_{s_2 \rightarrow 0} \frac{(0+s_2)^2 \frac{0 + s_2^2}{0 + s_2^2} - 0}{s_2} = 0$$

$$= \lim_{s_2 \rightarrow 0} \frac{s_2^2 - s_2^2}{s_2^2 + s_2^2} = -1$$

B. find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

$$\sin(xy) + \sin(yz) + \sin(zx) = 1 \quad \text{Implicit relation}$$

we will assume $z = f(x,y)$

xy - independent variable

Differentiating (1) w.r.t x partially, treating y as constant.

$$y \cos(xy) + y \cos(yz) \cdot \frac{dz}{dx} + \cos(zx) \frac{dz}{dx} = 0$$

can't treat z as constant (function of xy)

$$\frac{dz}{dx} [y \cos(yz) + x \cos(zy)] = - [y \cos(zy) + z \cos(zy)]$$

$$\frac{dz}{dx} = - \frac{[y \cos(zy) + z \cos(zy)]}{[y \cos(yz) + x \cos(zy)]}$$

Now
 $\sin(zy) + \sin(yz) + \sin(zx) = 1$

$$\frac{dz}{dy} = ?$$

$$\cos(zy) \cdot z + \cos(yz) \left(y \cdot \frac{dz}{dy} + z \right) + \cos(zx) \cdot x \cdot \frac{dz}{dy} = 0$$

$$\frac{dz}{dy} [y \cos(yz) + z \cos(zx)] = - [x \cos(zy) + z \cos(yz)]$$

$$\frac{dz}{dy} = - \frac{[x \cos(zy) + z \cos(yz)]}{[y \cos(yz) + x \cos(zx)]}$$

5. Also that you the surface $xyz^2 = c$ implicit.

$$\frac{dz}{dy} = -(x \log x)^{-1} \text{ at the point } x=y=z$$

$$xyz^2 = c$$

$$x \log x + y \log y + z \log z = \log c \quad (1)$$

$$\log c = \log 1$$

$$x=y=z$$

$$\log x + \log x + \log x = 0 + \log x + \log x + \log x = 0$$

$$\log x + \log x + \log x = 0$$

$$\frac{dz}{dx} = - \frac{(\log x + 1)}{(\log x + 1)} \quad (2)$$

$$\text{Similarly } \frac{dz}{dy} = ?$$

$$\frac{dz}{dy} = - \frac{(\log y + 1)}{(\log y + 1)} \quad (3)$$

$$\frac{dz}{dx} = - \frac{(\log x + 1)}{(\log x + 1)}$$

$$= - \frac{(\log x + 1)}{(\log x + 1)}$$

$$y = \text{const}$$

$$= - \frac{1}{(\log x + 1)} \cdot (1 + \log y)$$

$$= - (1 + \log y) \frac{1}{(\log x + 1)}$$

$$= - (1 + \log y) \cdot (-1) (1 + \log x)^{-1} \left(0 + \frac{1}{x} \cdot \frac{dz}{dx} \right)$$

$$= (1 + \log y) (1 + \log x)^{-1} \cdot \frac{1}{x} \frac{dz}{dx}$$

$$= \frac{(1 + \log y)}{(1 + \log x)^2} \cdot \frac{1}{x} \left[- \frac{(\log x + 1)}{(1 + \log x)^2} \right] \frac{dz}{dx}$$

$$= - \frac{(1 + \log y) (1 + \log x)}{x (1 + \log x)^3}$$

$$\text{at } x=y=z$$

$$= - \frac{(1 + \log x) (1 + \log x)}{x (1 + \log x)^3}$$

$$= - \frac{1}{x (1 + \log x)^2}$$

$$= - \frac{1}{x (1 + \log x)^2}$$

2. $f(x,y) = x \log y$. Check condition at (2,3)
 $f_x = \log y$ $f_x(2,3) = \log 3$

$f_y = \frac{x}{y}$ $f_y(2,3) = \frac{2}{3}$

$f_{xx} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial x} (\log y) = 0 \Rightarrow f_{xx}(2,3) = 0$

$f_{xy} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial y} \left(\frac{x}{y} \right) = f_{yx}(2,3) = -\frac{2}{9}$

$f_{yy} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial y} \left(\frac{x}{y} \right) = -\frac{x}{y^2} \Rightarrow f_{yy}(2,3) = -\frac{1}{3}$

$f_{xx} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = 0$

$f_{xy} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = 0$

$f_{yy} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial y} \left(\frac{x}{y} \right) = -\frac{x}{y^2} \Rightarrow f_{yy}(2,3) = -\frac{1}{3}$

$f_{xy} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial x} \left(\frac{x}{y} \right) = \frac{1}{y} \Rightarrow f_{xy}(2,3) = \frac{1}{3}$

Homogeneous function $f(x,y)$ is said to be homogeneous of degree n if it can be expressed as

$f(x,y) = x^n f\left(\frac{y}{x}\right)$

$f(x,y) = x^n f\left(\frac{y}{x}\right)$

$f(x,y) = x^n f\left(\frac{y}{x}\right)$

Q. $f(x,y) = \ln\left(\frac{y}{x}\right)$.

Sol: $f(x,y) = \ln\left(\frac{y}{x}\right)$ expand $\ln\left(\frac{y}{x}\right)$.

Homogeneous

Q. $f(x,y,z) = \frac{1}{x^2+y^2+z^2}$.

Sol: $n = 1$ homo, $d = 1$ homo

So, degree = $0 - 4 = -4$.
 So, f is homo of degree -4 .

Q. $f(x,y,z) = \sqrt{x^2+y^2+z^2}$.

Sol: degree = $\frac{1}{2} - 1 = -\frac{1}{2}$ homo.

Euler's theorem - (for homogeneous fun. only) -
 If $f(x,y)$ is a homogeneous function of degree n
 $\Rightarrow x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = n f$

The sum of the product of variable and its partial diff. equals to n times the fun. itself.

$f(x_1, x_2, \dots, x_n)$
 $x_1 \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + \dots + x_n \frac{\partial f}{\partial x_n} = n f(x_1, x_2, \dots, x_n)$

Ex: $f(x,y) = \frac{x^2 - 3xy}{x+y}$ Verify Euler's th. -

$$\text{Sol: } f(x,y) = \frac{x^2 - 3xy}{x+y} = x \left(\frac{x}{x+y} \right)$$

So, $f(x,y)$ is a homo. function of degree 1.

$$\text{Verify: } x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = 1 \cdot f$$

$$\frac{\partial f}{\partial x} = \frac{(x^2 - 3y)(x+y) - (x^2 - 3xy)(1+y)}{(x+y)^2}$$

$$= \frac{x^2 - 3xy - 3xy - 3y^2 - x^2 - 3xy}{(x+y)^2}$$

$$= \frac{x^2 - 3y^2 - 6xy}{(x+y)^2}$$

$$\frac{\partial f}{\partial y} = \frac{(0 - 3x)(x+y) - (x^2 - 3xy)(1+0)}{(x+y)^2}$$

$$= \frac{-3x^2 - 3xy - x^2 + 3xy}{(x+y)^2}$$

$$= \frac{-4x^2}{(x+y)^2}$$

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = \frac{x^2 - 3xy^2 + 2x^2y}{(x+y)^2} = \frac{3x^2y}{(x+y)^2}$$

$$= \frac{x^2 - 3xy^2 - 2x^2y}{(x+y)^2}$$

$$= \frac{x^2 - 3xy^2 - 2xy^2}{(x+y)^2} = \frac{(x^2 - 3xy^2)}{(x+y)^2} \cdot 1$$

taking $(x+y)$ common

Proved

Q. $Q(x,y) = \cos\left(\frac{x+y}{\sqrt{x+y}}\right)$, then prove that -

$$x \frac{\partial^2 Q}{\partial x^2} + y \frac{\partial^2 Q}{\partial y^2} = -\frac{1}{2} Q$$

$$\text{Sol: } z = \cos u = \frac{x+y}{\sqrt{x+y}}$$

this funct. is homo. of degree $1 - \frac{1}{2} = \frac{1}{2}$.

$$z = \frac{1}{2} \left(1 + \frac{y}{x} \right)$$

$$x^{1/2} \left(1 + \sqrt{\frac{y}{x}} \right)$$

$$= x^{1/2} \phi\left(\frac{y}{x}\right)$$

∴ z is a homo. funct. of degree $\frac{1}{2}$ in x and y .

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = \frac{1}{2} z$$

$$\frac{\partial z}{\partial x} = \frac{1}{2} \left(1 + \sqrt{\frac{y}{x}} \right)$$

$$z = \cos u$$

$$x (-\sin u) \frac{\partial u}{\partial x} + y (-\sin u) \frac{\partial u}{\partial y} = -\frac{1}{2} \cos u$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = -\frac{1}{2} \cos u = -\frac{1}{2} z$$

Proved

6. $\sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$

Let $z = \frac{x^2+y^2}{x+y}$ then

Sol- $z = \frac{x^2+y^2}{x+y}$

$z = \frac{x^2(1+\frac{y^2}{x^2})}{x(1+\frac{y}{x})}$

$= x \cdot \frac{1+\frac{y^2}{x^2}}{1+\frac{y}{x}}$

is form of degree 1

Now $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} \left(x \cdot \frac{1+\frac{y^2}{x^2}}{1+\frac{y}{x}} \right)$

$z = \frac{x^2+y^2}{x+y}$

$\frac{\partial z}{\partial x} = \frac{\cos u}{\sin u}$

$\frac{\partial z}{\partial y} = \frac{\cos u}{\sin u}$

So $\frac{\partial}{\partial x} \left(\frac{x^2+y^2}{x+y} \right) + \frac{\partial}{\partial y} \left(\frac{x^2+y^2}{x+y} \right) = \frac{\cos u}{\sin u}$

$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \frac{\cos u}{\sin u}$

$= \frac{\cos u}{\sin u}$

Ex 23 (11)

NON-CONTINUOUS, So, first principle should be used

All for the function $f(x,y) = \int_0^{x^2+y^2} \frac{1}{t^2+y^2} dt$, $(x,y) \neq (0,0)$

Show that $f_y = \frac{y}{x^2+y^2}$ at $(0,0)$

Sol- $f_x = \lim_{s \rightarrow 0} \frac{f(x+s, y) - f(x, y)}{s}$

$f_x(0,0) = \lim_{s \rightarrow 0} \frac{f(s, 0) - f(0,0)}{s}$

$= \lim_{s \rightarrow 0} \frac{(s^2)^2 \cdot 0 - 0}{(s^2)^2 + 0^2} = 0$

$f_y = \lim_{s \rightarrow 0} \frac{f(x, y+s) - f(x, y)}{s}$

$f_y(0,0) = \lim_{s \rightarrow 0} \frac{f(0, s) - f(0,0)}{s}$

$= \lim_{s \rightarrow 0} \frac{0 - 0}{s} = 0$

$f_x(0,y) = \lim_{s \rightarrow 0} \frac{f(s, y) - f(0, y)}{s}$

$= \lim_{s \rightarrow 0} \frac{s^2 \cdot y (s^2+y^2) - 0}{s^2+y^2} = 0$

$= \lim_{s \rightarrow 0} \frac{s y (s^2+y^2)}{(s^2+y^2)} = 0$

$f_y(x,0) = \lim_{s \rightarrow 0} \frac{f(x, s) - f(x,0)}{s}$

$= \lim_{s \rightarrow 0} \frac{s x^2 s y (s^2+y^2) - 0}{s^2+y^2} = 0 = \lim_{s \rightarrow 0} \frac{x^2 (s^2+y^2)}{(s^2+y^2)}$

$= \lim_{s \rightarrow 0} \frac{x^2 (1 - \frac{s^2}{1+s^2})}{1 + \frac{s^2}{1+s^2}}$

$$f_{xy}(0,0) = \left(\frac{\partial^2 f}{\partial x \partial y} \right)_{(0,0)}$$

$$= \lim_{s \rightarrow 0} \frac{f_y(s, 0) - f_y(0, 0)}{s}$$

$$= \lim_{s \rightarrow 0} \frac{s - 0}{s} = 1$$

$$f_{yx}(0,0) = \lim_{s \rightarrow 0} \frac{f_x(0, s) - f_x(0, 0)}{s}$$

$$= \lim_{s \rightarrow 0} \frac{0 - 0}{s} = 0$$

$$f_{xy} \neq f_{yx} \text{ at } (0,0)$$

OR: Also that $f_{xy} = f_{yx} \neq f_{yx}(0,0)$ when $f(x,y) = \log(x + \sqrt{y^2 + x^2})$.
New function is cont. point is excluded.

$$\frac{\partial f}{\partial x} = \frac{1}{(x + \sqrt{y^2 + x^2})} \left[1 + \frac{(0 + 2x)}{2\sqrt{y^2 + x^2}} \right]$$

$$= \frac{2(x + \sqrt{y^2 + x^2})}{(x + \sqrt{y^2 + x^2}) \cdot 2\sqrt{y^2 + x^2}}$$

$$= \frac{1}{\sqrt{y^2 + x^2}}$$

$$\frac{\partial f}{\partial y} = \frac{1}{(x + \sqrt{y^2 + x^2})} \left[0 + \frac{(2y)}{2\sqrt{y^2 + x^2}} \right]$$

$$= \frac{y}{(\sqrt{y^2 + x^2})^2}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{1}{\sqrt{y^2 + x^2}} \right)$$

$$= -\frac{1}{2} \frac{(2y)}{(y^2 + x^2)^{3/2}}$$

$$= -\frac{y}{(y^2 + x^2)^{3/2}}$$

$$\text{Now } \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{y}{(\sqrt{y^2 + x^2})^2} \right)$$

$$= \frac{y}{2x} \times \frac{(x + \sqrt{y^2 + x^2}) - y \cdot \frac{1}{2} \cdot \frac{2x}{\sqrt{y^2 + x^2}}}{(y^2 + x^2)^2} + \frac{(x + \sqrt{y^2 + x^2})}{2(y^2 + x^2)^{3/2}}$$

$$= -\frac{y}{2} \left[\frac{x + \sqrt{y^2 + x^2} - \frac{xy}{\sqrt{y^2 + x^2}}}{(y^2 + x^2)^2} + \frac{(x + \sqrt{y^2 + x^2})}{2(y^2 + x^2)^{3/2}} \right]$$

$$= -\frac{y}{2} \left[\frac{\sqrt{y^2 + x^2}(x + \sqrt{y^2 + x^2}) + x(x + \sqrt{y^2 + x^2})}{(y^2 + x^2)^{5/2}} \right] = -\frac{y}{2} \frac{(x^2 + y^2)^{3/2}}{(y^2 + x^2)^{5/2}}$$

$$\Rightarrow f_{xy} = f_{yx}$$

⇒ A function is homogeneous function of x, y of degree n .

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = nf \quad \text{--- (1)}$$

f = function of x^2y .

Method -
 $x^2 \frac{\partial f}{\partial x} + 2xy \frac{\partial f}{\partial x^2} + y^2 \frac{\partial f}{\partial y} = n(n-1)f$

And - Diff @ partially w.r.t x keeping y const

$$\frac{\partial f}{\partial x} + 2xy \frac{\partial f}{\partial x^2} + \frac{1-2f}{2x} + y \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = n \frac{\partial f}{\partial x}$$

cancel.

$$x \frac{\partial f}{\partial x} + \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial x^2} = n \frac{\partial f}{\partial x} \quad \text{--- (2) } \times x$$

Diff @ partially w.r.t y keeping x const

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + \frac{\partial f}{\partial y} = n \frac{\partial f}{\partial y} \quad \text{--- (3) } \times y$$

$$x \times \text{(2)} + y \times \text{(3)}$$

$$x^2 \frac{\partial f}{\partial x} + x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + y \frac{\partial f}{\partial y} + y^2 \frac{\partial f}{\partial y} + y \frac{\partial f}{\partial y} = n \frac{\partial f}{\partial x} + n \frac{\partial f}{\partial y}$$

$$x^2 \frac{\partial f}{\partial x} + y^2 \frac{\partial f}{\partial y} + \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = (n-1)x \frac{\partial f}{\partial x} + (n-1)y \frac{\partial f}{\partial y}$$

$$= (n-1)f$$

Q.31. If $u = x \tan\left(\frac{y}{x}\right) - y^2 \tan^2\left(\frac{y}{x}\right)$

find $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2}$

Sol.

Let $u = x \tan\left(\frac{y}{x}\right) - y^2 \tan^2\left(\frac{y}{x}\right)$

So, find is homogeneous of degree 2

$$-2x \cdot x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 2(2-2)u = 0$$

Q.33. If $\tan u = \frac{x^2+y^2}{x-y}$ then $x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} = 8 \sin 2u$

and $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = (1-4 \sin^2 u) 8 \sin 2u$

tan is a homogeneous funt. of degree 0.

Sol. $\tan u = x^2 \left[1 + \left(\frac{y}{x} \right)^2 \right] = x^2 \left(\frac{y}{x} \right)$ Degree = 3

$$x \left[1 - \left(\frac{y}{x} \right) \right]$$

Using Euler's th.

$$x \frac{\partial (\tan u)}{\partial x} + y \frac{\partial (\tan u)}{\partial y} = 3 \tan u$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} = 2 \tan u$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 8 \sin 2u \quad \text{--- (1)}$$

Can't apply directly Euler's th. as u is not homogeneous from tan u is homogeneous

Diff. Q. w.r.t x

$$1. \frac{\partial u}{\partial x} + x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial y \partial x} = 2 \cos \theta u \cdot \frac{\partial u}{\partial x}$$

$$\frac{\partial u}{\partial x} + x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial x \partial y} = 2 \cos \theta u \cdot \frac{\partial u}{\partial x} \quad (2)$$

Diff. Q. w.r.t y

$$x \frac{\partial^2 u}{\partial x \partial y} + 1. \frac{\partial u}{\partial y} + y \frac{\partial^2 u}{\partial y^2} = 2 \cos \theta u \cdot \frac{\partial u}{\partial y} \quad (3)$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} = 2 \cos \theta u \cdot \frac{\partial u}{\partial x}$$

$$x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial y^2} + x y \frac{\partial^2 u}{\partial x \partial y} + y \frac{\partial^2 u}{\partial x \partial y} = 2 \cos \theta u \cdot \frac{\partial u}{\partial x}$$

$$2 y \cos \theta u \cdot \frac{\partial u}{\partial y}$$

$$x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial y^2} + 2 x y \frac{\partial^2 u}{\partial x \partial y} = x \frac{\partial u}{\partial x} (2 \cos \theta u - 1)$$

$$+ y \frac{\partial u}{\partial y} (2 \cos \theta u - 1)$$

$$= (2 \cos \theta u - 1) \left[x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right]$$

$$= (2 \cos \theta u - 1) \cdot 2 \sin \theta u$$

$$= 2(1 - 4 \sin^2 \theta) \sin \theta u$$

Q. If $u = x^2 \left(\frac{y}{x} \right) + y \left(\frac{y}{x} \right)$

Find $\frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 0$

Sol - Individual terms are homogeneous but nothing is homogeneous

Let $u = x^2 \left(\frac{y}{x} \right) \cdot t = y \left(\frac{y}{x} \right)$

2-nd term of degree = 1, 1-nd term of degree = 0.

Now, $u = y^2 \left(\frac{y}{x} \right)$

Apply Euler's th. —

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 1.3$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 0 \quad (1)$$

$$t = y \left(\frac{y}{x} \right)$$

$$x \frac{\partial t}{\partial x} + y \frac{\partial t}{\partial y} = 0$$

$$x^2 \frac{\partial^2 t}{\partial x^2} + 2xy \frac{\partial^2 t}{\partial x \partial y} + y^2 \frac{\partial^2 t}{\partial y^2} = 0 \quad (2)$$

Adding (1) and (2), we get —

$$x^2 \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 t}{\partial x^2} \right] + 2xy \left[\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 t}{\partial x \partial y} \right] + y^2 \left[\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 t}{\partial y^2} \right] = 0$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} = 0$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 0$$

Total Differential coeff. =

If $u = f(x, y)$ is a funct. of 2 ord x and y are fund. of single variable t .



$\frac{du}{dt}$ - total diff. coeff.

$$\Rightarrow \frac{du}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt}$$

x is a fund. of single variable y

u is fund. of 2 variable

Let $u = x^2 y$
 $x = t$ and $y = t^2$

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt}$$

$$= 2xy \cdot 1 + 3y^2 \cdot 2t$$

$$= 2xy^3 + 6y^2 x^2 t$$

$$= 2t(t^2)^3 + 6t^2 \cdot t^2 \cdot t^2 +$$

$$= 2t^7 + 6t^7$$

$$= 8t^7$$

$$u = x^2 y^3$$

$$= t^2 (t^2)^3$$

$$= t^2 \cdot t^6 = t^8$$

$$\frac{du}{dt} = 8t^7$$

$u = f(x_1, x_2, \dots, x_n)$

and $x_1, x_2, \dots, x_n$ are fund. of single variable t

$$\frac{du}{dt} = \frac{\partial u}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial u}{\partial x_2} \frac{dx_2}{dt} + \dots + \frac{\partial u}{\partial x_n} \frac{dx_n}{dt}$$

Q935 -

(i) $u = f(x, y)$ and y is a fund. of x

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \frac{dy}{dx}$$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \frac{dy}{dx} \quad \text{--- (1)}$$

• x is a fund. of y

$$\frac{du}{dy} = \frac{\partial u}{\partial x} \frac{dx}{dy} + \frac{\partial u}{\partial y}$$

Also if we have an implicit relation,

$$f(x, y) = c$$

from (1)

$$0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx}$$

$$\frac{dy}{dx} = - \frac{\partial f / \partial x}{\partial f / \partial y}$$

Q.229

W. $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$

$x = t^2, y = te^t, z = te^{-t}$

$\frac{du}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt} + \frac{\partial u}{\partial z} \frac{dz}{dt}$

$\frac{du}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt} + \frac{\partial u}{\partial z} \frac{dz}{dt}$

$\frac{du}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt} + \frac{\partial u}{\partial z} \frac{dz}{dt}$

$W = y + yz + zx$

$\frac{\partial W}{\partial x} = y + z$

$\frac{\partial W}{\partial y} = 1 + z$

$\frac{\partial W}{\partial z} = y + x$

$\frac{dW}{dt} = \frac{\partial W}{\partial x} \frac{dx}{dt} + \frac{\partial W}{\partial y} \frac{dy}{dt} + \frac{\partial W}{\partial z} \frac{dz}{dt}$

$\frac{dW}{dt} = \frac{\partial W}{\partial x} \frac{dx}{dt} + \frac{\partial W}{\partial y} \frac{dy}{dt} + \frac{\partial W}{\partial z} \frac{dz}{dt}$

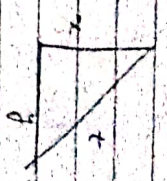
$\frac{dW}{dt} = \frac{\partial W}{\partial x} \frac{dx}{dt} + \frac{\partial W}{\partial y} \frac{dy}{dt} + \frac{\partial W}{\partial z} \frac{dz}{dt}$

$\frac{dW}{dt} = \frac{\partial W}{\partial x} \frac{dx}{dt} + \frac{\partial W}{\partial y} \frac{dy}{dt} + \frac{\partial W}{\partial z} \frac{dz}{dt}$

Which variable is regarded as constant -

$z = \frac{1}{2}xy$ — (1)

$y = \sqrt{t^2 - x^2}$
 $z = \frac{1}{2}x\sqrt{t^2 - x^2}$ — (2)



from (1)

$\left(\frac{\partial z}{\partial y}\right) = \frac{1}{2}x$ — (3) $[y \rightarrow \text{constant}]$

from (2)

$\left(\frac{\partial z}{\partial x}\right) = \frac{1}{2}\sqrt{t^2 - x^2} + \frac{1}{2}x \left[-\frac{1}{2}(t^2 - x^2)^{-1/2}(-2x) \right]$
 $\left(\frac{\partial z}{\partial x}\right) = \frac{1}{2}\sqrt{t^2 - x^2} - \frac{x^2}{2\sqrt{t^2 - x^2}}$ — (4) $[t \rightarrow \text{constant}]$

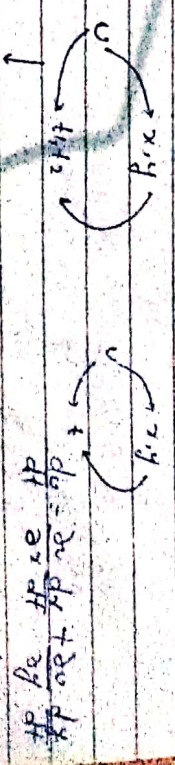
$(x,y) \rightarrow (x,0)$
 $(x,y,z) \rightarrow (x,0,0)$

changes of variables -

$qf u = f(x,y)$

$z = q(t_1, t_2)$

$u = q(t_1, t_2)$



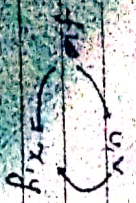
$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial t}$

$z \rightarrow \text{is a funct. of } \frac{\partial u}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial t}$

$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial t}$

Q. 29. If $u = f(x, y, z)$
 $u = \sqrt{x^2 + y^2 + z^2}$, $v = \cos^{-1}\left(\frac{y}{x}\right)$

then, S.T. $\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \frac{1}{x^2 + y^2} \left[(x^2 + y^2)^2 \left(\frac{\partial f}{\partial v}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2 \right]$



$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x}$

$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \left(\frac{x}{\sqrt{x^2 + y^2 + z^2}} \right) + \frac{\partial f}{\partial v} \left(\frac{-1}{1 + y^2/x^2} \cdot x \left(-\frac{y}{x^2} \right) \right)$

$\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2 + y^2 + z^2}} \cdot \frac{\partial f}{\partial u} + \frac{y}{x^2 + y^2} \cdot \frac{\partial f}{\partial v}$

So, $\left(\frac{\partial f}{\partial x}\right)^2 = \frac{x^2}{x^2 + y^2 + z^2} \left(\frac{\partial f}{\partial u}\right)^2 + \frac{y^2}{(x^2 + y^2)^2} \left(\frac{\partial f}{\partial v}\right)^2$

$+ \frac{2xy}{(x^2 + y^2)^{3/2}} \cdot \frac{\partial f}{\partial u} \cdot \frac{\partial f}{\partial v}$ (1)

$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y}$

$= \frac{\partial f}{\partial u} \left(\frac{y}{\sqrt{x^2 + y^2 + z^2}} \right) + \frac{\partial f}{\partial v} \left(\frac{-1}{1 + y^2/x^2} \cdot \left(\frac{1}{x} \right) \right)$

$\frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2 + y^2 + z^2}} \cdot \frac{\partial f}{\partial u} - \frac{\partial f}{x^2 + y^2}$

$\left(\frac{\partial f}{\partial y}\right)^2 = \frac{y^2}{x^2 + y^2 + z^2} \left(\frac{\partial f}{\partial u}\right)^2 + \frac{1}{(x^2 + y^2)^2} \left(\frac{\partial f}{\partial v}\right)^2$

$- \frac{2xy}{(x^2 + y^2)^{3/2}} \cdot \frac{\partial f}{\partial u} \cdot \frac{\partial f}{\partial v}$ (2)

Add (1) and (2)

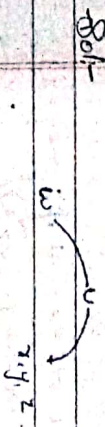
$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \frac{1}{x^2 + y^2} \left[(x^2 + y^2)^2 \left(\frac{\partial f}{\partial v}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2 \right]$

$\square$

Q. 30. If $u = \sin^{-1} v$, $v = \frac{(x^2 + y^2 + z^2)}{x^2 + y^2 + z^2}$, then

Show that -

$x \frac{\partial w}{\partial x} + y \frac{\partial w}{\partial y} + z \frac{\partial w}{\partial z} = \frac{w}{\sin w}$



$\frac{\partial w}{\partial x} = \frac{dw}{dv} \cdot \frac{\partial v}{\partial x} = \frac{1}{\sqrt{1-v^2}} \cdot \frac{2x}{x^2 + y^2 + z^2}$

$= \frac{1}{\sqrt{1-v^2}} \cdot \frac{2x}{x^2 + y^2 + z^2}$

$= \frac{1}{\sqrt{1-v^2}} \cdot \frac{2x}{x^2 + y^2 + z^2}$

$\frac{\partial w}{\partial y} = \frac{dw}{dv} \cdot \frac{\partial v}{\partial y} = \frac{1}{\sqrt{1-v^2}} \cdot \frac{2y}{x^2 + y^2 + z^2}$

$= \frac{1}{\sqrt{1-v^2}} \cdot \frac{2y}{x^2 + y^2 + z^2}$

$$\frac{\partial w}{\partial z} = \frac{1}{\sqrt{1-u^2}} \left[\frac{2xz + yz + z^2 - x^2 - y^2}{(1+y+z)^2} \right]$$

$$\text{Now } x \frac{\partial w}{\partial x} + y \frac{\partial w}{\partial y} + z \frac{\partial w}{\partial z}$$

$$= \frac{1}{\sqrt{1-u^2}} (2xy+z) \left[\frac{x(2^2+2xy+2xz-y^2-z^2) + y(2yz+2yz+y^2-z^2-z^2) + z(2xz+2yz+z^2-x^2-y^2)}{(1+y+z)^2} \right]$$

$$= \frac{1}{\sqrt{1-u^2}} (2xy+z) \left[\frac{x^3+2x^2y+2xz^2-xy^2-z^2}{(1+y+z)^2} + \frac{2xy^2+2yz^2+y^3-zy^2-yz^2}{(1+y+z)^2} + \frac{2xz^2+2yz^2+yz^2-z^2z-y^2z}{(1+y+z)^2} \right]$$

$$= \frac{1}{\sqrt{1-u^2}} (2xy+z) \left[\frac{x^3+y^3+z^3+x^2y+x^2z+xy^2+y^2z}{(1+y+z)^2} \right]$$

$$= \frac{1}{\sqrt{1-u^2}} (2xy+z) \left[\frac{z^3(1+y+z) + y^3(1+y+z)}{(1+y+z)^2} \right]$$

$$= \frac{1}{\sqrt{1-u^2}} (2xy+z) (x^2+y^2+z^2)$$

$$= \frac{u}{\sqrt{1-u^2}} \Rightarrow \frac{\partial h u}{\partial 1-u^2} = \frac{\sin u}{\cos u}$$

$$= \tan u$$

Can also apply Euler's thm

Q3

If the three thermodynamic variables P, V, T are connected by a relation $f(P, V, T) = 0$

$$\left(\frac{\partial f}{\partial T} \right)_P \left(\frac{\partial T}{\partial V} \right)_P \left(\frac{\partial V}{\partial P} \right)_T = -1$$

Sol. Diff. partially w.r.t T keeping V constant

$$\frac{\partial f}{\partial T} \frac{\partial T}{\partial T} + \frac{\partial f}{\partial V} \frac{\partial V}{\partial T} + \frac{\partial f}{\partial P} \frac{\partial P}{\partial T} = 0$$

$$\left(\frac{\partial f}{\partial T} \right)_V = - \frac{\partial f / \partial P}{\partial f / \partial V} \quad (2)$$

Diff. w.r.t V keeping P constant

$$\frac{\partial f}{\partial P} \frac{\partial P}{\partial V} + \frac{\partial f}{\partial V} \frac{\partial V}{\partial V} + \frac{\partial f}{\partial T} \frac{\partial T}{\partial V} = 0$$

$$\left(\frac{\partial T}{\partial V} \right)_P = - \frac{\partial f / \partial P}{\partial f / \partial T} \quad (3)$$

Diff. (1) w.r.t P keeping T constant

$$\frac{\partial f}{\partial P} \frac{\partial P}{\partial P} + \frac{\partial f}{\partial V} \frac{\partial V}{\partial P} + \frac{\partial f}{\partial T} \frac{\partial T}{\partial P} = 0 \quad (4)$$

$$\left(\frac{\partial V}{\partial P} \right)_T = - \frac{\partial f / \partial P}{\partial f / \partial V} \quad (4)$$

$$\text{Q12. If } z = \cos \theta, y = \sin \theta$$

Recall that

$$r^2 \frac{\partial^2 x}{\partial y^2} - 2xy \frac{\partial^2 x}{\partial x \partial y} + y^2 \frac{\partial^2 x}{\partial x^2} = \frac{\partial^2 x}{\partial x^2} + x \frac{\partial^2 x}{\partial x^2}$$

Sol. LHS $\rightarrow$ ✓

$$\text{RHS} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$= \frac{\partial}{\partial x} (\cos x) + \frac{\partial}{\partial y} (\sin y)$$

$$0 \rightarrow \text{const.}$$

$$\frac{\partial^2 u}{\partial x^2} = \cos x$$

$$\frac{\partial^2 u}{\partial y^2} = \sin y$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \cos x$$

$$\frac{\partial^2 u}{\partial y^2} = \sin y$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

Integrating the operator from 1st order diff. eq. for $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$ separately.

$$\frac{\partial^2 u}{\partial x^2} = \cos x$$

$$\frac{\partial^2 u}{\partial y^2} = \sin y$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \cos x$$

$$\frac{\partial^2 u}{\partial y^2} = \sin y$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \cos x$$

$$\frac{\partial^2 u}{\partial y^2} = \sin y$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \cos x$$

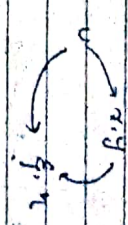
$$\frac{\partial^2 u}{\partial y^2} = \sin y$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \cos x$$

$$\frac{\partial^2 u}{\partial y^2} = \sin y$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$



$$\frac{\partial^2 u}{\partial x^2} = \cos x$$

$$\frac{\partial^2 u}{\partial y^2} = \sin y$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \cos x$$

$$\frac{\partial^2 u}{\partial y^2} = \sin y$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \cos x$$

$$\frac{\partial^2 u}{\partial y^2} = \sin y$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \cos x$$

$$\frac{\partial^2 u}{\partial y^2} = \sin y$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \cos x$$

$$\frac{\partial^2 u}{\partial y^2} = \sin y$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \cos x$$

$$\frac{\partial^2 u}{\partial y^2} = \sin y$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial^2 z}{\partial x^2} + x \frac{\partial^2 z}{\partial x \partial y} + y \frac{\partial^2 z}{\partial y^2} = 0 \quad (5)$$

$$\text{Let } \frac{\partial z}{\partial y} = \frac{\partial v}{\partial y} [e^{\sin y} + i e^{\cos y}]$$

$$+ \frac{\partial v}{\partial y} [e^{\sin y} + i e^{\cos y}]$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 v}{\partial y^2}$$

$$\text{Now } x = y = \frac{\partial v}{\partial y} \cos y \text{ and } y = \frac{\partial v}{\partial y} \sin y$$

$$(x-y)^2 = \frac{\partial^2 v}{\partial y^2} \cos^2 y - \frac{\partial^2 v}{\partial y^2} \sin^2 y = -e^{\sin y}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 v}{\partial y^2} \left[\frac{(x-y)^2}{2} + \frac{(1+y^2)}{2} \right] + \frac{\partial v}{\partial y} \left[\frac{y(1+x)}{2} + \frac{(1+y^2)}{2} \right]$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 v}{\partial x^2} \left[\frac{(x-y)^2}{2} + \frac{(1+y^2)}{2} \right] + \frac{\partial v}{\partial x} \left[\frac{y(1+x)}{2} + \frac{(1+y^2)}{2} \right]$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 v}{\partial x \partial y} [x+y] + \frac{\partial v}{\partial y} [y]$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 v}{\partial y^2} + \frac{\partial v}{\partial y}$$

$$\frac{\partial^2 z}{\partial y^2} = \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \left(\frac{\partial v}{\partial x} x + \frac{\partial v}{\partial y} y \right)$$

$$\frac{\partial^2 z}{\partial x^2} = \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + \frac{\partial v}{\partial y} \left(\frac{\partial v}{\partial x} x + \frac{\partial v}{\partial y} y \right)$$

$$\frac{\partial^2 z}{\partial x \partial y} = \left(\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 v}{\partial y \partial x} \right) + \frac{\partial v}{\partial y} \left[\frac{\partial v}{\partial x} x + \frac{\partial v}{\partial y} y \right]$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial v}{\partial y} \left[\frac{\partial v}{\partial x} x + \frac{\partial v}{\partial y} y \right]$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 v}{\partial y^2} + \frac{\partial v}{\partial y}$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}$$

APPLICATIONS

$$(x,y) \rightarrow (u,v)$$

$$= \iint f(x,y) dx dy$$

If $f(x,y)$ is a function of x and y

If u,v are functions of x and y independent variable x and y

$$\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} = \frac{\partial y}{\partial u} \frac{\partial x}{\partial v}$$

Then the determinant of transformation from u,v to x and y

$$\frac{\partial(x,y)}{\partial(u,v)} \rightarrow \text{notation} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = J(u,v)$$

If u,v,w are functions of x,y,z

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = J(u,v,w)$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{J(u,v,w)}$$

Let $x = r \cos \theta$, $y = r \sin \theta$
 $(x, y) = (r, \theta)$

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix}$$

$$= r \cos^2 \theta + r \sin^2 \theta$$

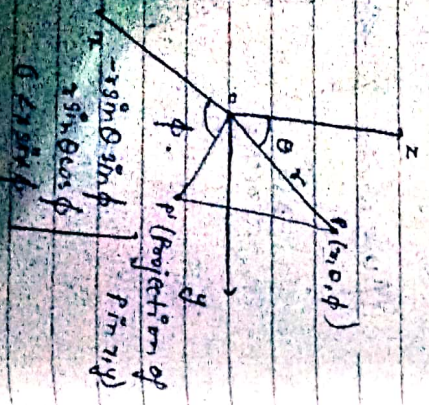
$= r$. (Remember this)

$$\iint f(x, y) dx dy = \iint f(r \cos \theta, r \sin \theta) \cdot r dr d\theta$$

Cartesian $\rightarrow$ Polar

Let $(x, y, z) = (r, \theta, \phi)$ in Cartesian $\rightarrow$ Spherical polar coordinate.
 $x = r \sin \theta \cos \phi$
 $y = r \sin \theta \sin \phi$
 $z = r \cos \theta$

$$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin \theta \cos \phi & -r \sin \theta \sin \phi & -r \cos \theta \sin \phi \\ \sin \theta \sin \phi & r \sin \theta \cos \phi & -r \cos \theta \cos \phi \\ \cos \theta & -r \sin \theta & 0 \end{vmatrix}$$



$$r^2 \begin{vmatrix} \sin \theta \cos \phi & \cos \theta \cos \phi & -\sin \theta \sin \phi \\ \sin \theta \sin \phi & \cos \theta \sin \phi & \sin \theta \cos \phi \\ \cos \theta & -\sin \theta & 0 \end{vmatrix}$$

$$= r^2 [\sin \theta \cos \phi (-\sin^2 \theta \sin^2 \theta \cos \phi) - \cos^2 \theta \cos \phi (-\cos^2 \theta \sin \theta \cos \phi) - \sin^2 \theta \sin \phi (-\sin^2 \theta \sin \phi - \cos^2 \theta \sin \phi)]$$

$$= r^2 [\sin \theta \cos \phi (\sin^2 \theta \cos \phi) - \cos^2 \theta \cos \phi (-\cos^2 \theta \sin \theta \cos \phi) - \sin^2 \theta \sin \phi (-\sin^2 \theta \sin \phi - \cos^2 \theta \sin \phi)]$$

$$= r^2 [\sin^3 \theta \cos^2 \phi + \cos^4 \theta \sin \theta \cos^2 \phi + \sin^3 \theta \sin^2 \phi + \sin \theta \cos^2 \theta \sin^2 \phi]$$

$$= r^2 [\sin^3 \theta + \cos^3 \theta \sin \theta]$$

Change of Variables -

$f(x, y, z)$ is a funⁿ of the independent variables x, y and z are funⁿ of 3 new variables u, v, w .

$$x = \phi(u, v, w)$$

$$y = \psi(u, v, w)$$

$$z = \chi(u, v, w)$$

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial u}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial v}$$

Cramer's Rule -

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 1$$

$$\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} - \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} - \frac{\partial f}{\partial z}$$

The determinant -

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial(x,y)}{\partial(u,v)}$$

is called Jacobian of transformation from xy to uv .

Similarly

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial v} = \frac{\partial f}{\partial z}$$

$$= \begin{vmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

and $\frac{\partial f}{\partial u} - \frac{\partial f}{\partial v} = \frac{\partial f}{\partial z}$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \end{vmatrix} = \frac{\partial(x,f)}{\partial(u,v)}$$

$$\frac{\partial f}{\partial x} = \frac{1}{J} \begin{vmatrix} \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{vmatrix}$$

Jacobian trans of xy to uv

Similarly, $\frac{\partial f}{\partial y} = \frac{-1}{J} \begin{vmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial z} \end{vmatrix}$

You can find second order partial derivative by using Jacobian transformation

$\#$ If $p(x,y,z)$ is a fcn of 3 independent variables and x,y,z are fcn of 3 independent variables.

$$x = p(u,v,w)$$

$$y = q(u,v,w)$$

$$z = r(u,v,w)$$

$$f(x,y,z)$$

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial u}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial v}$$

$$\frac{\partial f}{\partial w} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial w} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial w} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial w}$$

• $\frac{\partial f}{\partial x} = \frac{1}{J} \begin{vmatrix} \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$, where $J = \frac{\partial(x,y,z)}{\partial(u,v,w)}$

• $\frac{\partial f}{\partial y} = \frac{-1}{J} \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial z} \end{vmatrix}$

• $\frac{\partial f}{\partial z} = \frac{1}{J} \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{vmatrix}$

$$= \frac{1}{J} \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{vmatrix}$$

Q. $x = f(u, y), z = r \cos \theta$
 $y = r \sin \theta$

$\frac{1}{r^2} \left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 = \left(\frac{\partial f}{\partial z} \right)^2 + \frac{1}{r^2} \left(\frac{\partial f}{\partial \theta} \right)^2$



$\frac{\partial f}{\partial x} = \frac{1}{r} \frac{\partial f}{\partial (r \cos \theta)}$

$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$

$J = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix}$

$= r \cos^2 \theta + r \sin^2 \theta$
 $J = r$

$\frac{\partial}{\partial r} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial f}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial r}$

$= \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \end{vmatrix}$

$= \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \cos \theta & \sin \theta \end{vmatrix}$

$= \cos \theta \frac{\partial f}{\partial x} - \sin \theta \frac{\partial f}{\partial y}$

$\frac{\partial}{\partial \theta} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \theta}$

$= \frac{1}{r} \frac{\partial f}{\partial (r \cos \theta)} = \frac{1}{r} \frac{\partial f}{\partial x} \frac{\partial x}{\partial \theta} + \frac{1}{r} \frac{\partial f}{\partial y} \frac{\partial y}{\partial \theta}$

$= \frac{1}{r} \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix}$

$= \frac{1}{r} \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix}$

$= \frac{1}{r} \left[\cos \theta \cdot r \cos \theta + r \sin^2 \theta \right]$

$\left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 = \frac{1}{r^2} \left[r^2 \cos^2 \theta \left(\frac{\partial f}{\partial x} \right)^2 + r^2 \sin^2 \theta \left(\frac{\partial f}{\partial y} \right)^2 - \right.$

$\left. + 2 \cos \theta \sin \theta \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} \right]$

$= \frac{1}{r^2} \left[r^2 \cos^2 \theta \left(\frac{\partial f}{\partial x} \right)^2 + r^2 \sin^2 \theta \left(\frac{\partial f}{\partial y} \right)^2 + 2 \cos \theta \sin \theta \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} \right]$

$= \frac{1}{r^2} \left[r^2 \cos^2 \theta \left(\frac{\partial f}{\partial x} \right)^2 + r^2 \sin^2 \theta \left(\frac{\partial f}{\partial y} \right)^2 + 2 \cos \theta \sin \theta \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} \right]$

$= \left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2$

P.S.



Marking Rule:-

① $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial^2 xy}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}$

② $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0 \quad (a, b)$

③ $\text{Solve } x = \left(\frac{\partial^2 f}{\partial x^2} \right)_{(a,b)}, \quad y = \left(\frac{\partial^2 f}{\partial x \partial y} \right)_{(a,b)}$
 $t = \left(\frac{\partial^2 f}{\partial y^2} \right)_{(a,b)}$

④ If $t > 0 \Rightarrow (a, b)$ is a point max. or min.

(a) If $x < 0 \Rightarrow (a, b)$ is max.

(b) If $x > 0 \Rightarrow (a, b)$ is min.

(ii) If $t < 0 \Rightarrow (a, b)$ is a saddle point.

If $t = 0$, Nothing can be said and further investigation is needed.

Ex-2.4.

① $xy + \frac{1}{x} + \frac{3}{y}$

Sol:- $f(x, y) = xy + \frac{1}{x} + \frac{3}{y}$

$\frac{\partial f}{\partial x} = y - \frac{1}{x^2} = 0 \quad \text{--- (1)}$

$\frac{\partial f}{\partial y} = x - \frac{3}{y^2} = 0 \quad \text{--- (2)}$

$\Rightarrow x^2 y - 1 = 0 \Rightarrow y = \frac{1}{x^2} \quad \text{--- (3)}$

and $y^2 x - 3 = 0 \quad \text{--- (4)}$

$f_{\text{from (3)}} \quad y^2 \cdot x - 3 = 0$

$x^3 = 9^2$
 $x = 3, y = 1$

(3, 1).

$\frac{\partial^2 f}{\partial x^2} = \frac{2}{x^3}$

$u = \left(\frac{\partial^2 f}{\partial x^2} \right)_{(3,1)} = \frac{2}{3^3} = \frac{2}{27}$

$\frac{\partial^2 f}{\partial x \partial y} = 1 \Rightarrow v = \left(\frac{\partial^2 f}{\partial x \partial y} \right)_{(3,1)} = 1$

$\frac{\partial^2 f}{\partial y^2} = \frac{6}{y^3} \Rightarrow t = \left(\frac{\partial^2 f}{\partial y^2} \right)_{(3,1)} = 6$

At $(3, 1)$ is a point of max. or min.

$\Rightarrow (3, 1)$ is a point of min.

Q-349. $f(x,y) = 8(x^2 - y^2) - x^4 + y^4$

Sol- $\frac{\partial f}{\partial x} = 16x - 4x^3 = 0$
 $x = 0, \pm 1$

$\frac{\partial f}{\partial y} = -4y + 4y^3 = 0$
 $y = 0, \pm 1$

$(0,0), (0,-1), (1,0), (1,1), (-1,0), (-1,-1), (-1,1)$

(i). $(0,0)$
 $\frac{\partial^2 f}{\partial x^2} = 16 - 12x^2, x = \left(\frac{\partial^2 f}{\partial x^2}\right)_{(0,0)} = 16$

$\frac{\partial^2 f}{\partial x \partial y} = 0 \Rightarrow 9 = \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(0,0)} = 0$

$\frac{\partial^2 f}{\partial y^2} = -4 + 12y^2 \Rightarrow -4 = \left(\frac{\partial^2 f}{\partial y^2}\right)_{(0,0)} = -4$

$\Delta f - \Delta^2 = -16 < 0$
 $(0,0) \rightarrow$ Saddle points

Q-30 $\rightarrow 5, 6, 8, 14, 15, 19 \rightarrow$ check it's a hand.

Q Given $f(x,y,z) = 5xyz$
 $2+2y+4z$

Find the value of xyz for which $f(x,y,z)$ is a max. subject to the cond. $xyz=8$

Sol- $xyz=8$
 $x = \frac{8}{yz}$

$f(x,y) = \frac{5yz \times 8}{yz} = \frac{40}{yz}$
 $z + 2y + y \times \frac{8}{yz}$

$f(x,y) = \frac{40yz}{yz^2 + y^2z + 32}$

$F = \frac{1}{f}$ for max. then, F min.

Sol, $F = \frac{yz^2 + y^2z + 32}{40yz}$

$F = \frac{z}{40} + \frac{y}{20} + \frac{32}{40yz}$

$\frac{\partial F}{\partial x} = \frac{1}{40} + 0 + \frac{1}{40yz} \cdot \frac{-32}{y^2} \cdot \frac{1}{z} = 0$
 $\Rightarrow z^2y = 32 \text{ --- (1)}$

$\frac{\partial F}{\partial y} = \frac{1}{20} - \frac{32}{40} \cdot \frac{1}{yz^2} = 0 \Rightarrow 32yz^2 = 32 \text{ --- (2)}$

Sol. (1) and (2)
 $z = y, y = 2, z = 4$
 $(4, 2, 1)$

$$x = \left(\frac{\partial^2 F}{\partial x^2} \right)_{(y,z)} = \frac{1}{80}$$

$$s = \left(\frac{\partial^2 F}{\partial x \partial y} \right)_{(y,z)} = \frac{1}{80}$$

$$t = \left(\frac{\partial^2 F}{\partial y^2} \right)_{(y,z)} = \frac{1}{20}$$

$$80 \times 20 - 9^2 = \frac{1}{80 \times 20} - \frac{1}{80 \times 80}$$

$$= \frac{1}{80} \left(\frac{1}{20} - \frac{1}{80} \right)$$

$$= \frac{1}{80} \times \frac{3}{80} = \frac{3}{6400} > 0$$

$\nabla f(y,z)$ is max. or min.

(y,z) is point of min. for $F(x,y,z)$.

$\nabla f(y,z)$ is point of max. for $F = \frac{1}{F}$.

39. $f(x,y) = e^{2x+y}$ about $(0,0)$.

40. $f(x,y) = 8x(x+y)$ about $(0,0)$ } upto third term.

41. $f(x,y) = \sin(x+y)$, $f(0,0) = 0$
Taylor series about $(0,0)$ is -

$$f(x,y) = f(0,0) + (x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}) f \Big|_{(0,0)} + \frac{1}{2!} (x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y})^2 f \Big|_{(0,0)} + \frac{1}{3!} (x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y})^3 f \Big|_{(0,0)} + \dots$$

$$+ \frac{1}{3!} (x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y})^3 f \Big|_{(0,0)} + \dots$$

$$f_x = \cos(x+y), f_x(0,0) = 1$$

$$f_y = \cos(x+y), f_y(0,0) = 1$$

$$f_{xx} = -\sin(x+y), f_{xx}(0,0) = 0$$

$$f_{xy} = -\sin(x+y), f_{xy}(0,0) = 0$$

$$f_{yy} = -\sin(x+y), f_{yy}(0,0) = 0$$

$$f_{xxx} = -\cos(x+y), f_{xxx}(0,0) = -1$$

$$f(x,y) = f(0,0) + (x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}) f \Big|_{(0,0)} + \frac{1}{2!} (x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y})^2 f \Big|_{(0,0)} + \frac{1}{3!} (x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y})^3 f \Big|_{(0,0)} + \dots$$

$$+ \frac{1}{3!} (x^3 f_{xxx} + 3x^2 y f_{xxy} + 3xy^2 f_{xyx} + y^3 f_{yyy}) \Big|_{(0,0)} + \dots$$

$$f_{xyy} = -\cos(x+y), f_{xyy}(0,0) = -1$$

$$f_{yyy} = -\cos(x+y), f_{yyy}(0,0) = -1$$

$$f_{xyy} = -\cos(x+y), f_{xyy}(0,0) = -1$$

$$f_{yyy} = -\cos(x+y), f_{yyy}(0,0) = -1$$

$$\begin{aligned}
 f(x,y) &= 0 + (x+2y)^{1/2} (0+0+0) \\
 &+ \frac{1}{3} (-x^3 - 6x^2y + 12xy^2 + 8y^3) \\
 &= x + 2y + \frac{1}{3} (-8y^3 - x^3 - 6x^2y + 12xy^2) \\
 &= x + 2y - \frac{1}{3} (x^3 + 8y^3 + 6x^2y - 12xy^2) \\
 &= (x+2y) - \frac{1}{3} (x+2y)^3
 \end{aligned}$$

MAXIMA AND MINIMA — — local extrema values

Let $f(x,y)$ be defined on a region R containing the point (a,b) . then

- ① $f(a,b)$ is a local maximum (relative) maximum value of f if: $f(a,b) \geq f(x,y)$ for all domain points (x,y) in an open disk centered at (a,b) .
- ② $f(a,b)$ is a local (relative) minimum value of f if $f(a,b) \leq f(x,y)$ for all domain points (x,y) in an open disk centered at (a,b) .

Critical points —

An interior point of the domain of a function $f(x,y)$ where both f_x and f_y are zero or where one or both of f_x and f_y do not exist is a critical point of f .

Saddle Point —

A differentiable function $f(x,y)$ has a saddle point at a critical point (a,b) if in every open disk centered at (a,b) there are domain points (x,y) where $f(x,y) > f(a,b)$ and domain points (x,y) where $f(x,y) < f(a,b)$.

Second Derivative test —

Suppose $f(x,y)$ and its first and second partial derivatives are continuous through out a disk centered at (a,b) and that $f_x(a,b) = f_y(a,b) = 0$.

Then —

- ① f has local (relative) maximum at (a,b) if $f_{xx}f_{yy} > 0$ and $f_{xx} > 0$ at (a,b) .

② f has local (relative) minimum at (a,b) if $f_{xx} > 0$ and $f_{xy} = f_{yx} = 0$ at (a,b) .

③ f has a saddle point at (a,b) if $f_{xx} f_{yy} = f_{xy}^2 < 0$ at (a,b) .

④ f is inconclusive at (a,b) if $f_{xy} = 0$ at (a,b) .

If $f_{xx} = f_{yy} = 0 = f_{xy}$,
 • local maxima if $x < 0$, $at = s^2 > 0$
 minima if $x > 0$, $at = s^2 > 0$
 saddle point if $at = s^2 < 0$.

⑤ Find the point on the surface $z^2 = xy + 1$ nearest to the origin (find point of minima for this surface).

$D^2 = x^2 + y^2 + z^2$
 New $z^2 = xy + 1$

So $D^2 = x^2 + y^2 + xy + 1$

$\frac{\partial}{\partial x}$

$F = x^2 + y^2 + xy + 1$

$\frac{\partial F}{\partial x} = 2x + y = 0$ — (1)

$\frac{\partial F}{\partial y} = 2y + x = 0$ — (2)

Solving (1) and (2)

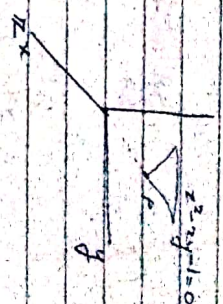
$x = y = 0$
 $\Rightarrow z = \pm 1$

Now $D^2 = \begin{pmatrix} \frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x \partial y} \\ \frac{\partial^2 F}{\partial y \partial x} & \frac{\partial^2 F}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

$S = \begin{pmatrix} \frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x \partial y} \\ \frac{\partial^2 F}{\partial y \partial x} & \frac{\partial^2 F}{\partial y^2} \end{pmatrix} = 1$

$t = \begin{pmatrix} \frac{\partial^2 F}{\partial y^2} & \frac{\partial^2 F}{\partial y \partial x} \\ \frac{\partial^2 F}{\partial x \partial y} & \frac{\partial^2 F}{\partial x^2} \end{pmatrix} = 2$

So Point of max. is min. $x > 0$.



14. In a plane triangle ABC find the max. value of $\cos A \cos B \cos C$
 $f(A, B, C) = \cos A \cos B \cos C - (A+B)$
 $= -\cos A \cos B \cos (A+B)$

$$\frac{\partial f}{\partial A} = -\cos B / -\cos A \sin(A+B) + \cos(A+B)(-\sin A) = 0$$

$$\Rightarrow \cos B \sin(A+B) = 0$$

$$\frac{\partial f}{\partial B} = -\cos A / -\sin(A+B) \cos B + \cos(A+B)(-\sin B) = 0$$

$$\Rightarrow \cos A \sin(A+B) = 0$$

Case i) $\cos B \sin(A+B) = 0$
 $\& \cos A \sin(A+B) = 0$

Case ii) $\sin(A+B) = 0$
 $\therefore B = \pi/2$
 $\& \cos A = 0 \Rightarrow A = \pi/2$
 not possible.

Case iii) $\sin(A+B) = 0$
 $\therefore B = \pi/2$
 $\& \cos A = 0, A = \pi/2$
 not possible.

Case iv) $\sin(A+B) = 0 \& \sin(A+B) = 0$
 $\& \cos A = 0, A = \pi/2$
 $\& \cos B = 0, B = \pi/2$
 $A+B = \pi$

$A+B = \pi/2$

* $\theta = \pi/3$.
 check $(\pi/3, \pi/3, \pi/3)$ is max.

Q. Expand $f(x, y)$ in the increasing powers of $x-a, y-b$
 expand $f(x, y)$ at the point (a, b) in Taylor series
 same.

$$f(x, y) = f(a, b) + \left[(x-a) \frac{\partial f}{\partial x} + (y-b) \frac{\partial f}{\partial y} \right]_{(a,b)} + \frac{1}{2!} \left[(x-a)^2 \frac{\partial^2 f}{\partial x^2} + 2(x-a)(y-b) \frac{\partial^2 f}{\partial x \partial y} + (y-b)^2 \frac{\partial^2 f}{\partial y^2} \right]_{(a,b)} + \dots$$

$f =$	(a, b)	upto at least 3 terms.
$\frac{\partial f}{\partial x}$		
$\frac{\partial^2 f}{\partial x^2}$		
$\frac{\partial^2 f}{\partial x \partial y}$		
$\frac{\partial^2 f}{\partial y^2}$		

At $(x, y) \Rightarrow$ At point (a, b) .
 3rd term.