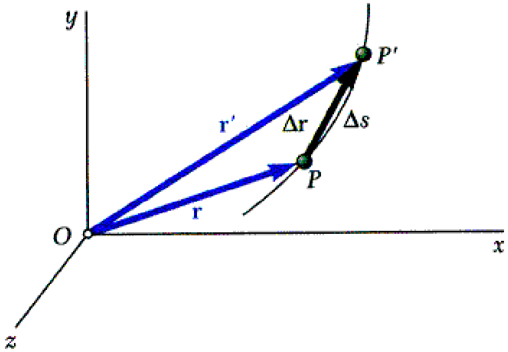
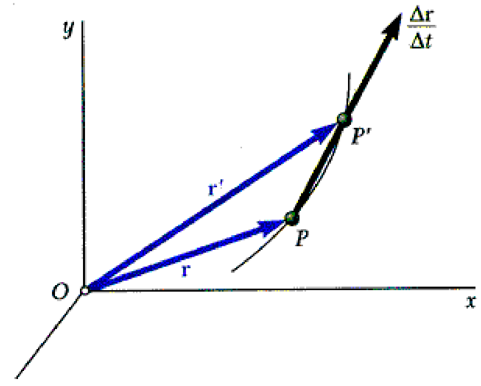
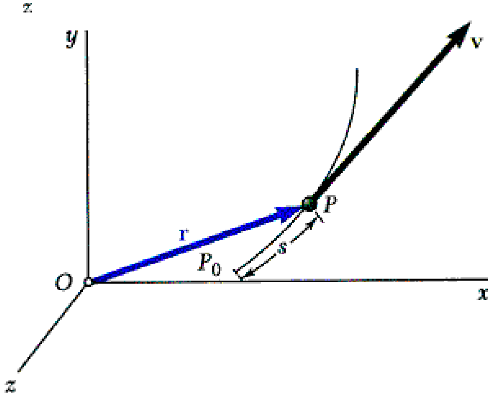
Contents

Curvilinear Motion: Position, Velocity & Acceleration







- Particle moving along a curve other than a straight line is in *curvilinear motion*.
- *Position vector* of a particle at time *t* is defined by a vector between origin *O* of a fixed reference frame and the position occupied by particle.
- Consider particle which occupies position *P* defined by $\vec{r}$ at time *t* and *P'* defined by $\vec{r}\,'$ at $t + \Delta t$,

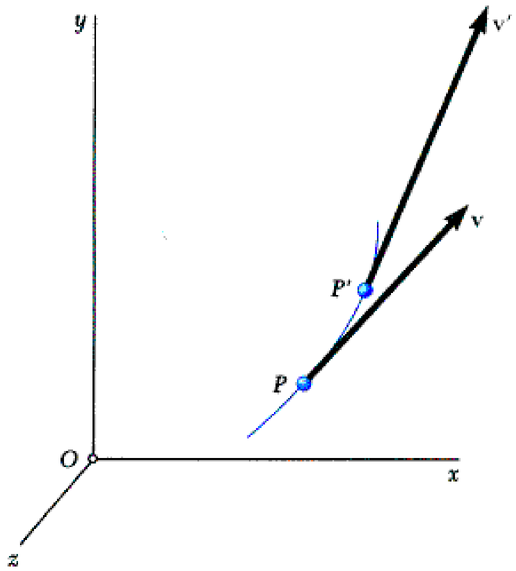
$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

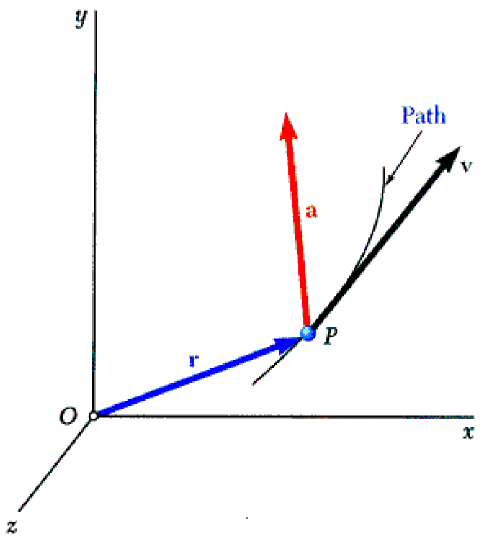
$=$ instantaneous velocity (vector)

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$$

$=$ instantaneous speed (scalar)

Curvilinear Motion: Position, Velocity & Acceleration





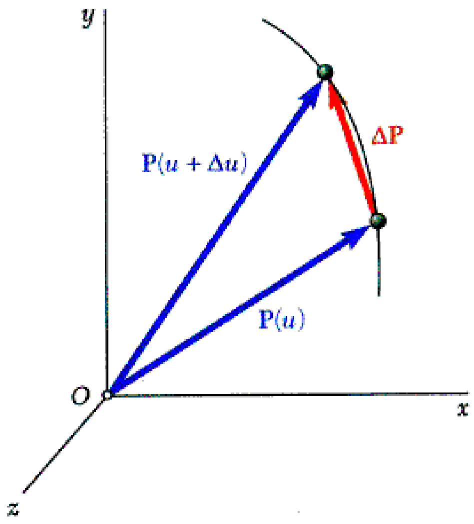
- Consider velocity $\vec{v}$ of particle at time t and velocity $\vec{v}'$ at $t + \Delta t$,

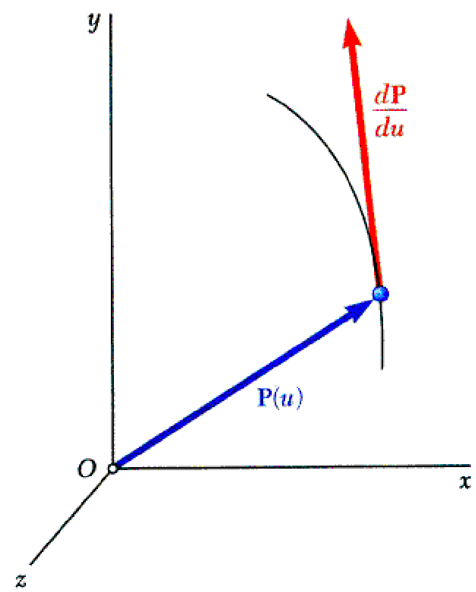
$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$$

$$= \text{instantaneous acceleration (vector)}$$

- In general, acceleration vector is not tangent to particle path and velocity vector.

Derivatives of Vector Functions





- Let $\vec{P}(u)$ be a vector function of scalar variable u,

$$\frac{d\vec{P}}{du} = \lim_{\Delta u \to 0} \frac{\Delta \vec{P}}{\Delta u} = \lim_{\Delta u \to 0} \frac{\vec{P}(u+\Delta u) - \vec{P}(u)}{\Delta u}$$

- Derivative of vector sum,

$$\frac{d\left(\vec{P}+\vec{Q}\right)}{du} = \frac{d\vec{P}}{du} + \frac{d\vec{Q}}{du}$$

- Derivative of product of scalar and vector functions,

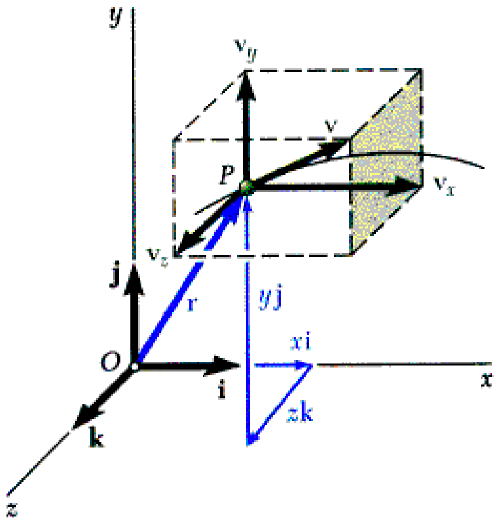
$$\frac{d\left(f\vec{P}\right)}{du} = \frac{df}{du}\vec{P} + f\frac{d\vec{P}}{du}$$

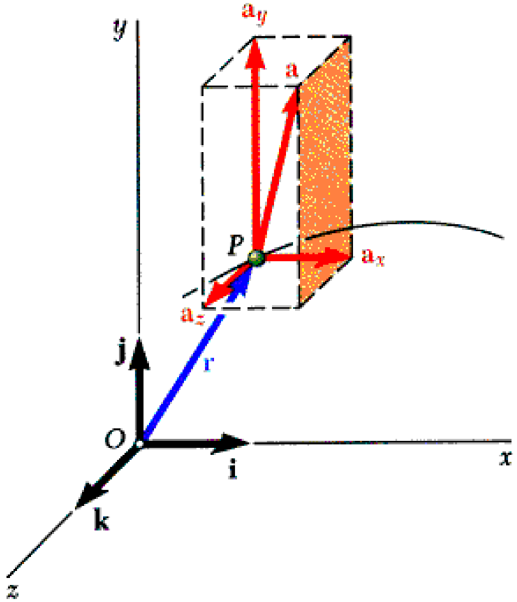
- Derivative of *scalar product* and *vector product*,

$$\frac{d\left(\vec{P}\bullet\vec{Q}\right)}{du} = \frac{d\vec{P}}{du}\bullet\vec{Q} + \vec{P}\bullet\frac{d\vec{Q}}{du}$$

$$\frac{d\left(\vec{P}\times\vec{Q}\right)}{du} = \frac{d\vec{P}}{du}\times\vec{Q} + \vec{P}\times\frac{d\vec{Q}}{du}$$

Rectangular Components of Velocity & Acceleration





- When position vector of particle *P* is given by its rectangular components,

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

- Velocity vector,

$$\vec{v} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k} = \dot{x}\vec{i} + \dot{y}\vec{j} + \dot{z}\vec{k}$$

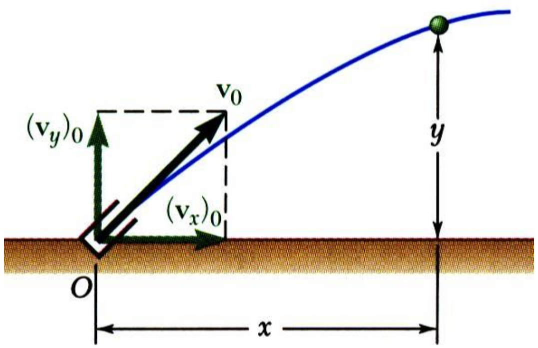
$$= v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$$

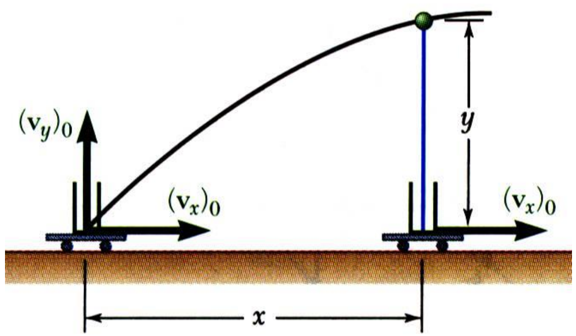
- Acceleration vector,

$$\vec{a} = \frac{d^2x}{dt^2}\vec{i} + \frac{d^2y}{dt^2}\vec{j} + \frac{d^2z}{dt^2}\vec{k} = \ddot{x}\vec{i} + \ddot{y}\vec{j} + \ddot{z}\vec{k}$$

$$= a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$$

Rectangular Components of Velocity & Acceleration





- Rectangular components particularly effective when component accelerations can be integrated independently, e.g., motion of a projectile,

$$a_x = \ddot{x} = 0 \qquad a_y = \ddot{y} = -g$$

with initial conditions,

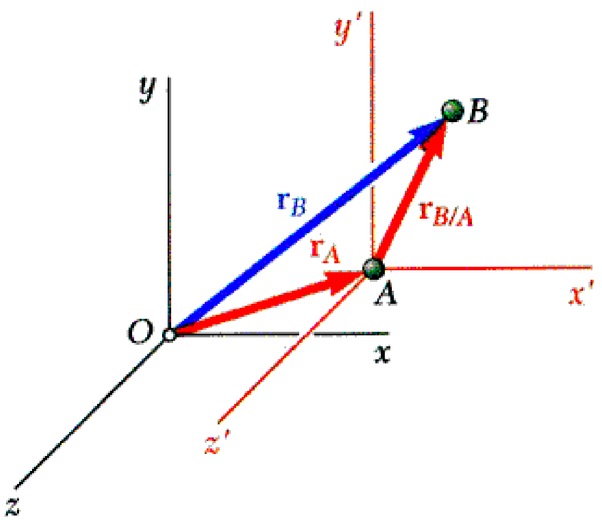
$$x_0 = y_0 = 0 \qquad (v_x)_0, (v_y)_0$$

Integrating twice yields

$$v_x = (v_x)_0 \quad v_y = (v_y)_0 - gt$$
$$x = (v_x)_0 t \quad y = (v_y)_0 y - \tfrac{1}{2} g t^2$$

- Motion in horizontal direction is uniform.
- Motion in vertical direction is uniformly accelerated.
- Motion of projectile could be replaced by two independent rectilinear motions.

Motion Relative to a Frame in Translation



- Designate one frame as the *fixed frame of reference*. All other frames not rigidly attached to the fixed reference frame are *moving frames of reference*.
- Position vectors for particles *A* and *B* with respect to the fixed frame of reference *Oxyz* are $\vec{r}_A$ and $\vec{r}_B$.
- Vector $\vec{r}_{B/A}$ joining *A* and *B* defines the position of *B* with respect to the moving frame *Ax'y'z'* and

$$\vec{r}_B = \vec{r}_A + \vec{r}_{B/A}$$

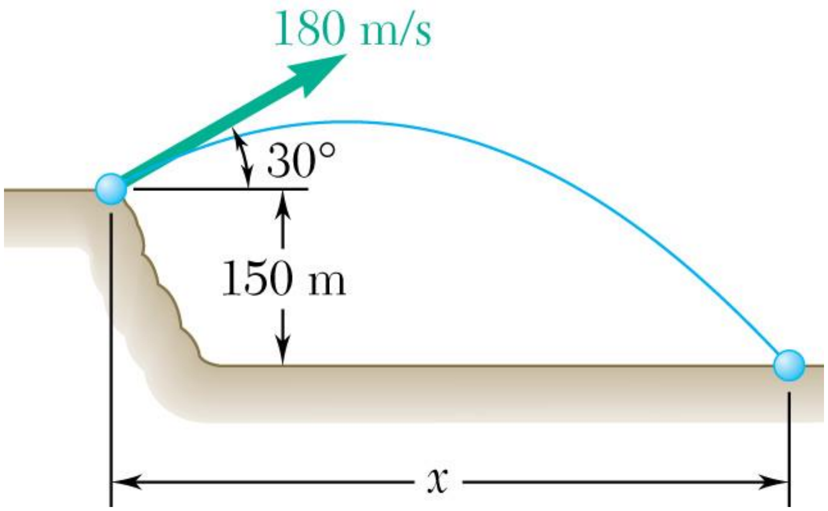
- Differentiating twice,

$$\vec{v}_B = \vec{v}_A + \vec{v}_{B/A} \qquad \vec{v}_{B/A} = \text{velocity of } B \text{ relative to } A.$$

$$\vec{a}_B = \vec{a}_A + \vec{a}_{B/A} \qquad \vec{a}_{B/A} = \text{acceleration of } B \text{ relative to } A.$$

- Absolute motion of *B* can be obtained by combining motion of *A* with relative motion of *B* with respect to moving reference frame attached to *A*.

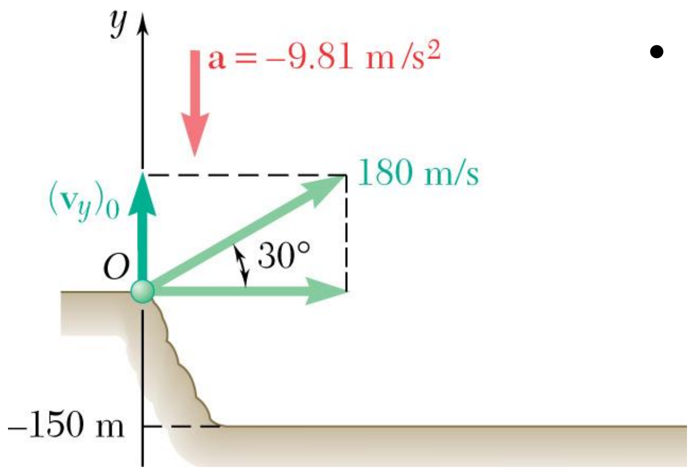
Sample Problem 11.7



A projectile is fired from the edge of a cliff with an initial velocity of 180m/s with 30° angle with the horizontal. Neglecting air resistance, find:

(a) the horizontal distance to the point where the projectile hits ground *(b)* the highest elevation above ground reached by the projectile.

Sample Problem 11.7



SOLUTION:

- For the vertical motion substitute the acceleration and initial velocity into the equations for uniformly accelerated motion.

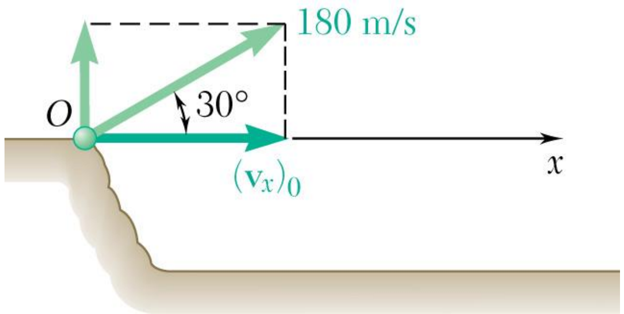
$$(v_y)_0 = (180\,\text{m/s})\sin 30 = 90\ \text{m/s}$$

$$a = -9.81\,\text{m/s}^2$$

$$v_y = (v_y)_0 + at \quad \Rightarrow \quad v_y = 90 - 9.81t$$

$$y = (v_y)_0 t + \frac{1}{2}at^2 \quad \Rightarrow \quad y = 90t - 4.9t^2$$

$$v_y^2 = (v_y)_0^2 + 2ay \quad \Rightarrow \quad v_y^2 = 8100 - 19.62y$$



- For the horizontal motion substitute the initial velocity into the uniform motion equation.

$$(v_x)_0 = (180\,\text{m/s})\cos 30 = 155.9\ \text{m/s}$$

$$x = (v_x)_0 t \quad \Rightarrow \quad x = 155.9t$$

Sample Problem 11.7

- Solve for the time needed to hit the ground then substitute to find the horizontal distance.

$$y = -150 \text{ m}$$

$$-150 = 90t - 4.9t^2$$

$$t^2 - 18.37t - 30.6 = 0$$

$$\Rightarrow t = 19.91 \text{ s}$$

$$x = 155.9(19.91)$$

$$\boxed{x = 3100 \text{ m}}$$

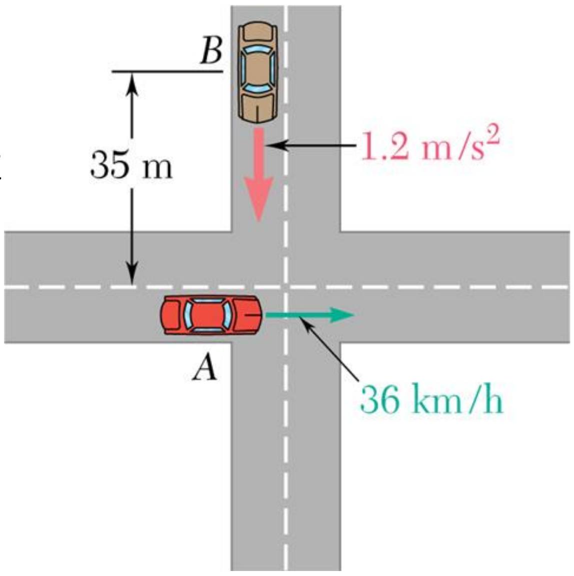
- Highest elevation reached when vertical velocity is zero.

$$0 = 8100 - 19.62y$$

$$\Rightarrow y = 413 \text{ m}$$

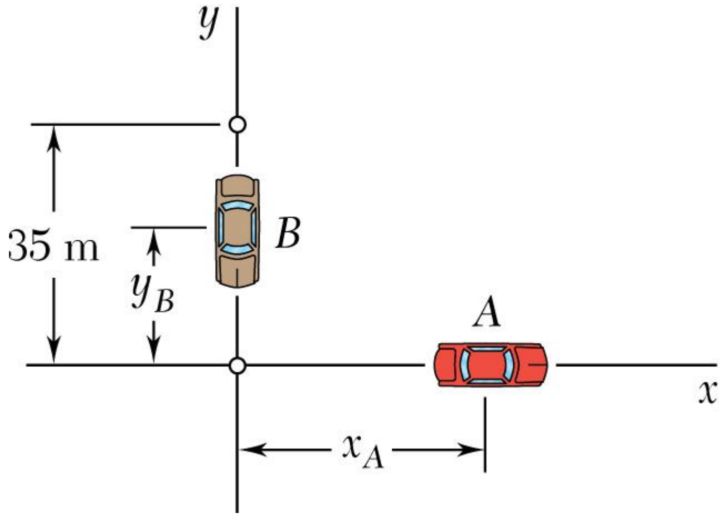
$$\boxed{\text{Highest Elevation } = 150 + 413 = 563 \text{ m}}$$

Sample Problem 11.9

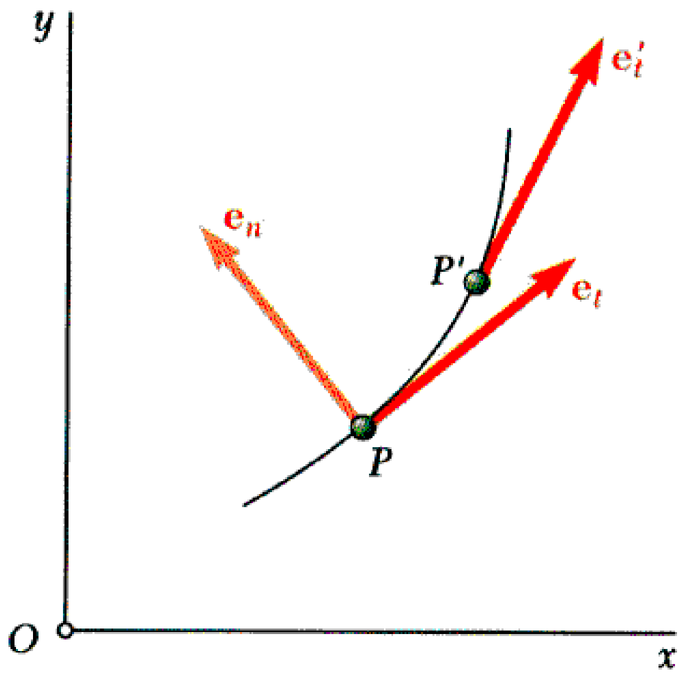


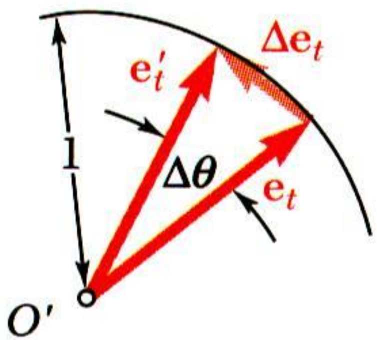
Car A is traveling east at a constant speed of 36km/h. As car A crosses the intersection, car B starts from rest 35m north of the intersection and moves south with a constant acceleration of 1.2m/s^2 .

Find: the position, velocity and acceleration of B relative to A 5s after A crosses the intersection.



Tangential and Normal Components





- Velocity vector of particle is tangent to path of particle. In general, acceleration vector is not. Wish to express acceleration vector in terms of tangential and normal components.

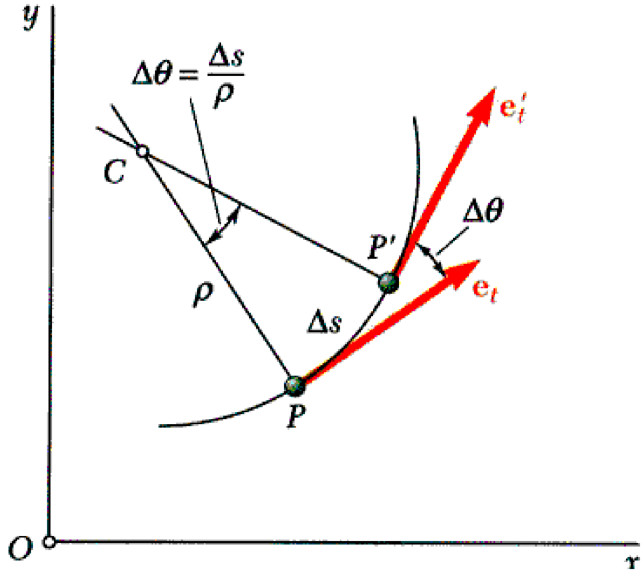
- $\vec{e}_t$ and $\vec{e}_t'$ are tangential unit vectors for the particle path at P and P'. When drawn with respect to the same origin, $\Delta\vec{e}_t = \vec{e}_t' - \vec{e}_t$ and $\Delta\theta$ is the angle between them.

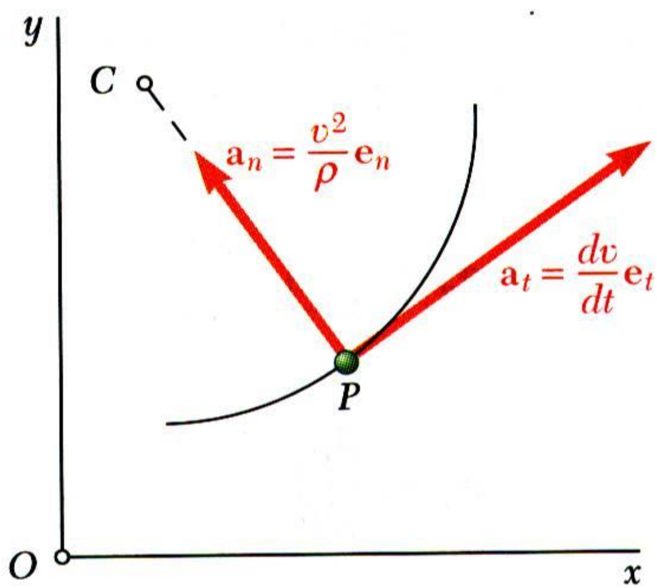
$$\Delta e_t = 2\sin(\Delta\theta/2)$$

$$\lim_{\Delta\theta\to 0}\frac{\Delta\vec{e}_t}{\Delta\theta} = \lim_{\Delta\theta\to 0}\frac{\sin(\Delta\theta/2)}{\Delta\theta/2}\vec{e}_n = \vec{e}_n$$

$$\vec{e}_n = \frac{d\vec{e}_t}{d\theta}$$

Tangential and Normal Components





- With the velocity vector expressed as $\vec{v} = v\vec{e}_t$ the particle acceleration may be written as

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv}{dt}\vec{e}_t + v\frac{d\vec{e}_t}{dt} = \frac{dv}{dt}\vec{e}_t + v\frac{d\vec{e}}{d\theta}\frac{d\theta}{ds}\frac{ds}{dt}$$

but

$$\frac{d\vec{e}_t}{d\theta} = \vec{e}_n \qquad \rho\, d\theta = ds \qquad \frac{ds}{dt} = v$$

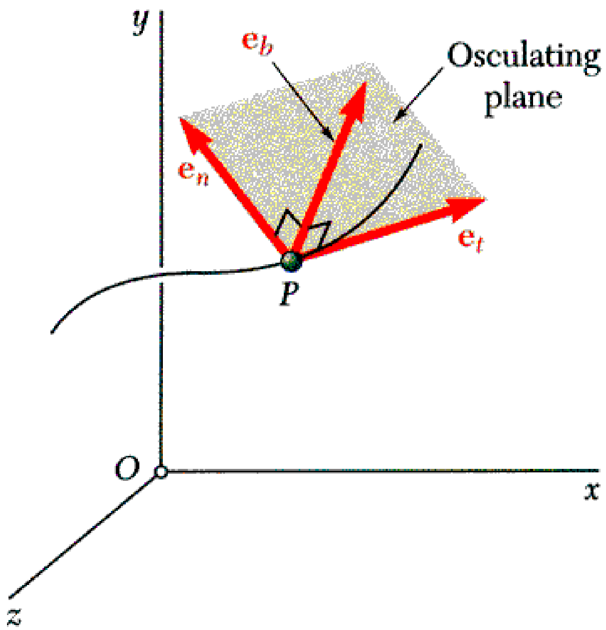
After substituting,

$$\vec{a} = \frac{dv}{dt}\vec{e}_t + \frac{v^2}{\rho}\vec{e}_n \qquad a_t = \frac{dv}{dt} \quad a_n = \frac{v^2}{\rho}$$

- Tangential component of acceleration reflects change of speed and normal component reflects change of direction.

- Tangential component may be positive or negative. Normal component always points toward center of path curvature.

Tangential and Normal Components



- Relations for tangential and normal acceleration also apply for particle moving along space curve.

$$\vec{a} = \frac{dv}{dt}\vec{e}_t + \frac{v^2}{\rho}\vec{e}_n \qquad a_t = \frac{dv}{dt} \quad a_n = \frac{v^2}{\rho}$$

- Plane containing tangential and normal unit vectors is called the *osculating plane*.

- Normal to the osculating plane is found from

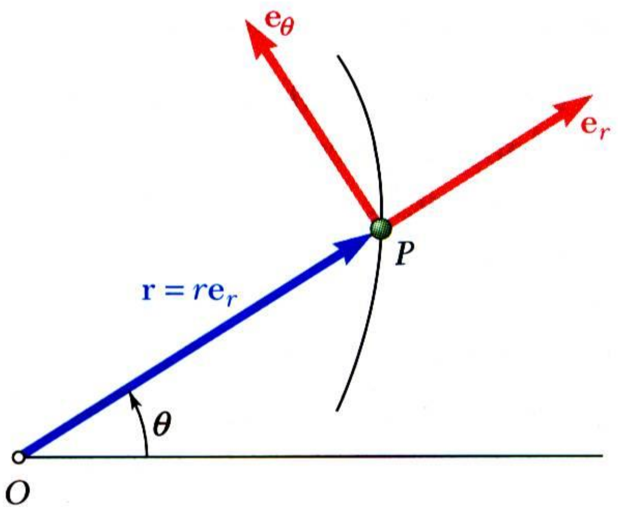
$$\vec{e}_b = \vec{e}_t \times \vec{e}_n$$

$$\vec{e}_n = principal\ normal$$

$$\vec{e}_b = binormal$$

- Acceleration has no component along binormal.

Radial and Transverse Components



$$\vec{r} = r\vec{e}_r$$

$$\frac{d\vec{e}_r}{d\theta} = \vec{e}_\theta \qquad \frac{d\vec{e}_\theta}{d\theta} = -\vec{e}_r$$

$$\frac{d\vec{e}_r}{dt} = \frac{d\vec{e}_r}{d\theta}\frac{d\theta}{dt} = \vec{e}_\theta \frac{d\theta}{dt}$$

$$\frac{d\vec{e}_\theta}{dt} = \frac{d\vec{e}_\theta}{d\theta}\frac{d\theta}{dt} = -\vec{e}_r \frac{d\theta}{dt}$$

- When particle position is given in polar coordinates, it is convenient to express velocity and acceleration with components parallel and perpendicular to *OP*.

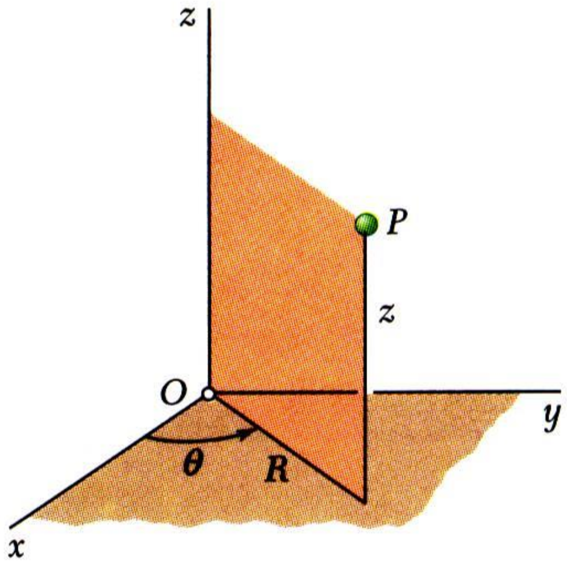
- The particle velocity vector is

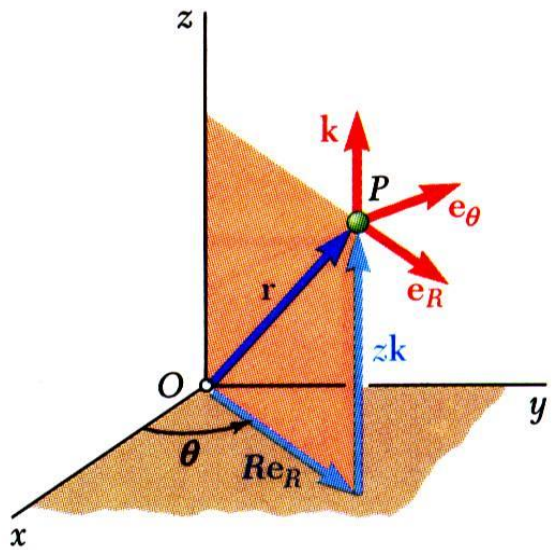
$$\vec{v} = \frac{d}{dt}\left(r\vec{e}_r\right) = \frac{dr}{dt}\vec{e}_r + r\frac{d\vec{e}_r}{dt} = \frac{dr}{dt}\vec{e}_r + r\frac{d\theta}{dt}\vec{e}_\theta$$
$$= \dot{r}\,\vec{e}_r + r\dot{\theta}\,\vec{e}_\theta$$

- Similarly, the particle acceleration vector is

$$\vec{a} = \frac{d}{dt}\left(\frac{dr}{dt}\vec{e}_r + r\frac{d\theta}{dt}\vec{e}_\theta\right)$$
$$= \frac{d^2r}{dt^2}\vec{e}_r + \frac{dr}{dt}\frac{d\vec{e}_r}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\vec{e}_\theta + r\frac{d^2\theta}{dt^2}\vec{e}_\theta + r\frac{d\theta}{dt}\frac{d\vec{e}_\theta}{dt}$$
$$= \left(\ddot{r} - r\dot{\theta}^2\right)\vec{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\vec{e}_\theta$$

Radial and Transverse Components





- When particle position is given in cylindrical coordinates, it is convenient to express the velocity and acceleration vectors using the unit vectors $\vec{e}_R, \vec{e}_\theta$, and $\vec{k}$.

- Position vector,

$$\vec{r} = R\vec{e}_R + z\vec{k}$$

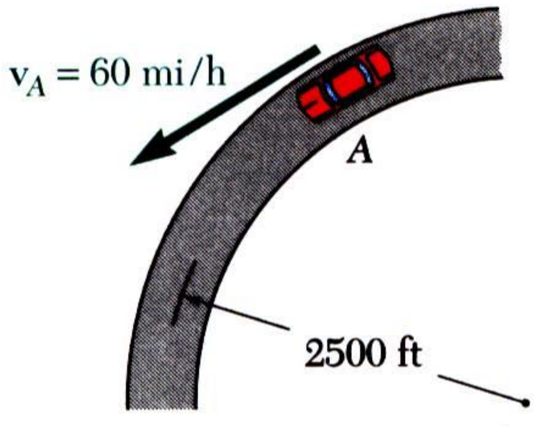
- Velocity vector,

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{R}\vec{e}_R + R\dot{\theta}\vec{e}_\theta + \dot{z}\vec{k}$$

- Acceleration vector,

$$\vec{a} = \frac{d\vec{v}}{dt} = \left(\ddot{R} - R\dot{\theta}^2\right)\vec{e}_R + \left(R\ddot{\theta} + 2\dot{R}\dot{\theta}\right)\vec{e}_\theta + \ddot{z}\vec{k}$$

Sample Problem 11.10



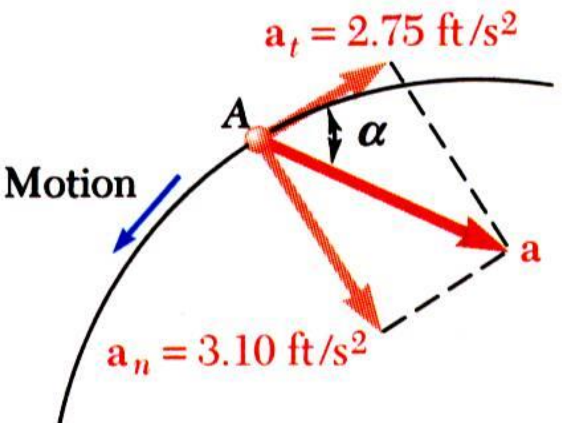
A motorist is traveling on curved section of highway at 60 mph. The motorist applies brakes causing a constant deceleration rate.

Knowing that after 8 s the speed has been reduced to 45 mph, determine the acceleration of the automobile immediately after the brakes are applied.

SOLUTION:

- Calculate tangential and normal components of acceleration.
- Determine acceleration magnitude and direction with respect to tangent to curve.

Sample Problem 11.10



$$60\,\text{mph} = 88\,\text{ft/s}$$

$$45\,\text{mph} = 66\,\text{ft/s}$$

SOLUTION:

- Calculate tangential and normal components of acceleration.

$$a_t = \frac{\Delta v}{\Delta t} = \frac{(66-88)\,\text{ft/s}}{8\,\text{s}} = -2.75\,\frac{\text{ft}}{\text{s}^2}$$

$$a_n = \frac{v^2}{\rho} = \frac{(88\,\text{ft/s})^2}{2500\,\text{ft}} = 3.10\,\frac{\text{ft}}{\text{s}^2}$$

- Determine acceleration magnitude and direction with respect to tangent to curve.

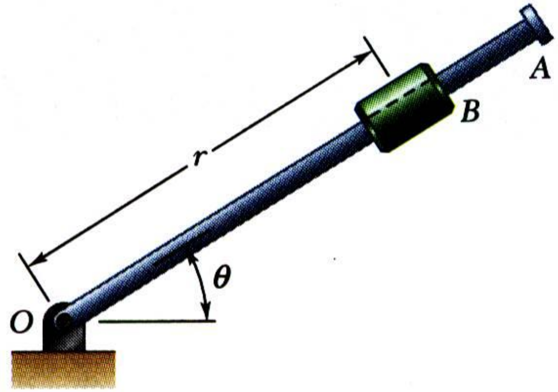
$$a = \sqrt{a_t^2 + a_n^2} = \sqrt{(-2.75)^2 + 3.10^2}$$

$$a = 4.14\,\frac{\text{ft}}{\text{s}^2}$$

$$\alpha = \tan^{-1}\frac{a_n}{a_t} = \tan^{-1}\frac{3.10}{2.75}$$

$$\alpha = 48.4^\circ$$

Sample Problem 11.12



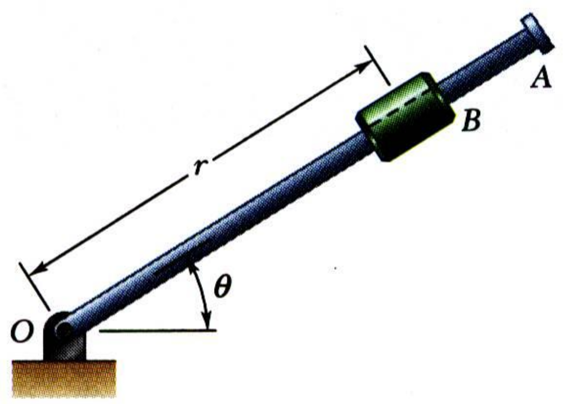
Rotation of the arm about O is defined by $\theta = 0.15t^2$ where θ is in radians and t in seconds. Collar B slides along the arm such that $r = 0.9 - 0.12t^2$ where r is in meters.

After the arm has rotated through 30°, determine *(a)* the total velocity of the collar, *(b)* the total acceleration of the collar, and *(c)* the relative acceleration of the collar with respect to the arm.

SOLUTION:

- Evaluate time t for $\theta = 30°$.
- Evaluate radial and angular positions, and first and second derivatives at time t.
- Calculate velocity and acceleration in cylindrical coordinates.
- Evaluate acceleration with respect to arm.

Sample Problem 11.12



SOLUTION:

- Evaluate time t for $\theta = 30^{o}$.

$$\theta = 0.15t^2$$

$$= 30° = 0.524\,\text{rad} \qquad t = 1.869\,\text{s}$$

- Evaluate radial and angular positions, and first and second derivatives at time t.

$$r = 0.9 - 0.12t^2 = 0.481\,\text{m}$$

$$\dot{r} = -0.24t = -0.449\,\text{m/s}$$

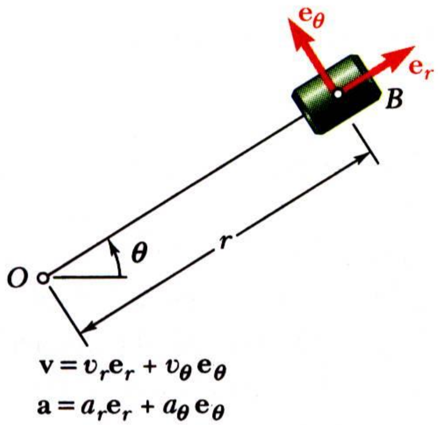
$$\ddot{r} = -0.24\,\text{m/s}^2$$

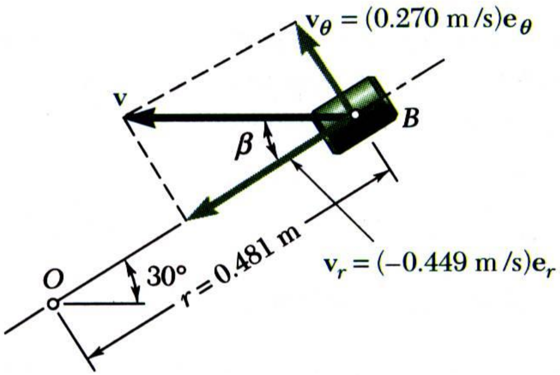
$$\theta = 0.15t^2 = 0.524\,\text{rad}$$

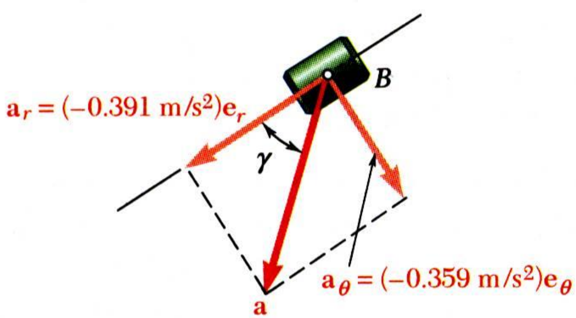
$$\dot{\theta} = 0.30t = 0.561\,\text{rad/s}$$

$$\ddot{\theta} = 0.30\,\text{rad/s}^2$$

Sample Problem 11.12







- Calculate velocity and acceleration.

$$v_r = \dot{r} = -0.449\,\mathrm{m/s}$$

$$v_\theta = r\dot{\theta} = (0.481\,\mathrm{m})(0.561\,\mathrm{rad/s}) = 0.270\,\mathrm{m/s}$$

$$v = \sqrt{v_r^2 + v_\theta^2} \qquad \beta = \tan^{-1}\frac{v_\theta}{v_r}$$

$$\boxed{v = 0.524\,\mathrm{m/s} \qquad \beta = 31.0^\circ}$$

$$a_r = \ddot{r} - r\dot{\theta}^2$$

$$= -0.240\,\mathrm{m/s^2} - (0.481\,\mathrm{m})(0.561\,\mathrm{rad/s})^2$$

$$= -0.391\,\mathrm{m/s^2}$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

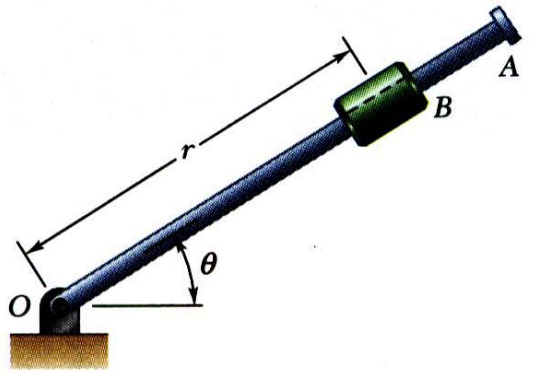
$$= (0.481\,\mathrm{m})\left(0.3\,\mathrm{rad/s^2}\right) + 2(-0.449\,\mathrm{m/s})(0.561\,\mathrm{rad/s})$$

$$= -0.359\,\mathrm{m/s^2}$$

$$a = \sqrt{a_r^2 + a_\theta^2} \qquad \gamma = \tan^{-1}\frac{a_\theta}{a_r}$$

$$\boxed{a = 0.531\,\mathrm{m/s} \qquad \gamma = 42.6^\circ}$$

Sample Problem 11.12



- Evaluate acceleration with respect to arm.

  Motion of collar with respect to arm is rectilinear and defined by coordinate *r*.

$$a_{B/OA} = \ddot{r} = -0.240\,\mathrm{m/s^2}$$