Ex. 12

Q7. A body falls from rest in a liquid whose density is one-fifth that of the body. If the liquid offers resistance proportional to the ~~body~~ velocity and the velocity approaches a limiting value of 10 meters per second, find the distance fallen in 10 seconds.

Soln The equation of the motion of the body is

$$m\frac{dv}{dt} = mg - \frac{1}{4}mg - mkv$$

or
$$\frac{dv}{dt} + kv = \frac{3}{4}g \quad \text{--- (1)}$$

The initial conditions are $v=0$ when $t=0$.

where mg, weight acting downwards.

$\frac{1}{4}mg$, upthrust acting upwards.

mkv, resistance acting upwards.

$\therefore$ The Laplace transform of equation (1) is

$$s\bar{v} + k\bar{v} = \frac{3}{4}g/s$$

$$\bar{v} = \frac{3g}{4}\Big/ s(s+k) = \frac{3g}{4k}\left(\frac{1}{s} - \frac{1}{s+k}\right) \quad \text{--- (2)}$$

Taking inverse L.T. of (2) we get.

Hence $$v = \frac{3g}{4k}\left(1 - e^{-kt}\right) \quad \text{--- (3)}$$

when $s \to \infty$, $v \to \frac{3g}{4k}$, given that the limiting velocity is 10 meters per second. so

$$\frac{3g}{4k} = 10 \quad \text{or} \quad k = \frac{3g}{40}$$

Equation (3) becomes

$$\frac{dx}{dt} = \frac{3g}{4\times \frac{3g}{40_{10}}}\left(1 - e^{-kt}\right) = 10\left(1 - e^{-kt}\right) \quad \text{--- (4)}$$

and the initial conditions is $x = 0$ when $t = 0$.

Laplace transform of equation (4) is

$$s\bar{x} = 10\left(\frac{1}{s} - \frac{1}{s+k}\right)$$

i.e. $$\bar{x} = 10\left(\frac{1}{s^2} - \frac{1}{s(s+k)}\right) = 10\left[\frac{1}{s^2} - \frac{1}{k}\left(\frac{1}{s} - \frac{1}{s+k}\right)\right]$$

Taking inverse L.T. we get

$$x = 10\left[t - \frac{1}{k}\left(1 - e^{-kt}\right)\right]$$

$$x = 10\left[t - \frac{40}{3g}\left(1 - e^{-\frac{3g}{40}t}\right)\right]$$

Put. $t = 10$ and $g = 9.8\ m/sec^2$ we get

$$x = 10\left[10 - \frac{40}{3 \times 9.8}\left(1 - e^{-\frac{3 \times 9.8}{404} \times 10}\right)\right]$$

$$x = ? \quad (\text{Simplify})$$

Q12. A voltage Ee^{-at} is applied at $t=0$ to a circuit of inductance L and resistance R. Show that the current at time t is

$$\frac{E}{R - aL}\left(e^{-at} - e^{-\frac{R}{L}t}\right).$$

Soln. Let I be the current in the circuit. Then the potential difference across the inductance L is $L\frac{dI}{dt}$ and across the resistance R is RI

$\therefore$ The equation of the electric circuit is

$$L\frac{dI}{dt} + RI = Ee^{-at} \quad \text{——} (1)$$

Laplace transform of equation (1) is

$$(Ls + R)\bar{I} = \frac{E}{s+a}$$

$$\bar{I} = \frac{E}{(Ls+R)(s+a)} = \frac{E/L}{(s + R/L)(s+a)} \quad \text{——} (2)$$

$$\bar{I} = \frac{E/L}{\frac{R}{L} - a}\left(\frac{1}{s+a} - \frac{1}{s+R/L}\right)$$

$$\bar{I} = \frac{E}{R-aL}\left(\frac{1}{s+a} - \frac{1}{s+R/L}\right) \text{ --- (3)}$$

Taking inverse L.T. of (3) we get.

$$I = \frac{E}{R-aL}\left(e^{-at} - e^{-\frac{R}{L}t}\right).$$

Q14. An alternating voltage $E \sin \omega t$ is applied at $t=0$ to a circuit of inductance L and resistance R. If the initial current be zero, show that the current at time t is

$$\frac{E}{\sqrt{R^2+L^2\omega^2}}\left[e^{-\frac{R}{L}t}\sin\gamma + \sin(\omega t - \gamma)\right],$$

where $\tan \gamma = \frac{L\omega}{R}$.

Soln. The equation of the electric circuit is

$$L\frac{dI}{dt} + RI = E \sin \omega t \text{ --- (1)}$$

where $L\frac{dI}{dt}$ is the potential difference across the inductance L

RI " " " the resistance R

I be the current in the circuit.

Taking L.T. of equation (1) we get -

$$(Ls+R)\bar{I} = \frac{E\omega}{s^2+\omega^2}$$

$$\bar{I} = \frac{E\omega}{(Ls+R)(s^2+\omega^2)} = \frac{\omega E/L}{(s+R/L)(s^2+\omega^2)}$$

$$= \frac{E\omega/L}{\omega^2 + R^2/L^2}\left(\frac{1}{s+R/L} - \frac{s-R/L}{s^2+\omega^2}\right) \text{ by inspection}$$

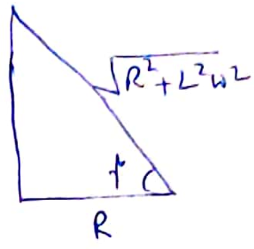
$$\bar{I} = \frac{LE\omega}{L^2\omega^2+R^2}\left(\frac{1}{s+R/L} - \left(\frac{s-R/L}{s^2+\omega^2}\right)\right) \quad \text{--- (2)}$$

$$\left(\frac{s}{s^2+\omega^2} - \frac{R/L}{s^2+\omega^2}\right)$$

Taking inverse L.T. of (2) we get

$$I = \frac{LE\omega}{L^2\omega^2+R^2}\left(e^{-\frac{R}{L}t} - \cos\omega t + \frac{R}{L\omega}\sin\omega t\right)$$

$\tan\varphi = \frac{L\omega}{R}$ (given)



$$\therefore \quad \sin\varphi = \frac{L\omega}{\sqrt{R^2+L^2\omega^2}} \Rightarrow L\omega = \sqrt{R^2+L^2\omega^2}\,\sin\varphi$$

$$\cos\varphi = \frac{R}{\sqrt{R^2+L^2\omega^2}} \Rightarrow R = \sqrt{R^2+L^2\omega^2}\,\cos\varphi$$

$$I = \frac{E\cancel{L\omega}\sqrt{R^2+L^2\omega^2}\cdot\sin\varphi}{R^2+L^2\omega^2}\left(e^{-\frac{R}{L}t} - \cos\omega t + \frac{\cos\varphi}{\sin\varphi}\cdot\sin\omega t\right)$$

$$= \frac{E}{\sqrt{R^2+L^2\omega^2}}\left[e^{-\frac{R}{L}t}\sin\varphi - \cos\omega t\sin\varphi + \cos\varphi\sin\omega t\right]$$

$$= \frac{E}{\sqrt{R^2+L^2\omega^2}}\left[e^{-\frac{R}{L}t}\sin\varphi + \sin(\omega t-\varphi)\right].$$

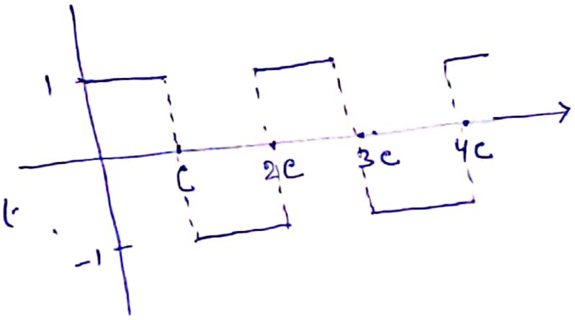
Laplace Transform of a periodic functions

Definition A function $f(x)$ is said to be periodic, if and only if $f(x+p) = f(x)$ — ① is true for some value of p and every value of x. The smallest positive value of p for which equation ① is true for every value of x will be called the periodic function of period p.

Laplace transform of the square wave function

Consider the square wave function with period $2c$. Then its Laplace transform is equal to

$$\int_0^c e^{-st}\cdot 1\,dt + \int_c^{2c} e^{-st}(-1)\,dt + \int_{2c}^{3c} e^{-st}\cdot 1\,dt + \int_{3c}^{4c} e^{-st}(-1)\,dt + \cdots$$



$$= \int_0^c e^{-st}\,dt - \int_c^{2c} e^{-st}\,dt + \int_{2c}^{3c} e^{-st}\,dt - \int_{3c}^{4c} e^{-st}\,dt + \cdots$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^c - \left[\frac{e^{-st}}{-s}\right]_c^{2c} + \left[\frac{e^{-st}}{-s}\right]_{2c}^{3c} - \left[\frac{e^{-st}}{-s}\right]_{3c}^{4c} + \cdots$$

$$= \frac{1}{s}\left[\left(1 - e^{-cs}\right) - \left(e^{-cs} - e^{-2cs}\right) + \left(e^{-2cs} - e^{-3cs}\right) - \left(e^{-3cs} - e^{-4cs}\right) + \cdots\right]$$

$$= \frac{1}{s}\left[1 - 2e^{-cs} + 2e^{-2cs} - 2e^{-3cs} + 2e^{-4cs} - \cdots\right]$$

$$= \frac{1}{s}\left[\frac{2}{1+e^{-cs}} - 1\right] = \frac{1}{s}\left[\frac{1-e^{-cs}}{1+e^{-cs}}\right] = \frac{1}{s}\left[\frac{e^{cs/2} - e^{-cs/2}}{e^{cs/2} + e^{-cs/2}}\right] = \frac{1}{s}\tanh\frac{cs}{2}.$$

Ex 1: Show that the Laplace transform of a periodic function $f(t)$ of period c is

$$L\{f(t)\} = \frac{1}{1-e^{-cs}} \int_0^c e^{-st} \cdot f(t)\,dt$$

Soln: By definition of L.T.

$$\cancel{\int_0^\infty e^{-st} f(t)\,dt = \int_0^c e^{-st} f(t)\,dt + \int_c^{2c} e^{-st} f(t)\,dt + \int_{2c}^{3c} e^{-st} f(t)\,dt + \dots \quad (1)}$$

~~Now put $t = u + nc$ we get~~ -

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^c e^{-st} f(t)\,dt + \int_c^\infty e^{-st} f(t)\,dt \quad \text{— (1)}$$

In the second integral of equation (1), put $t = u + c \therefore dt = du$

$$\therefore L\{f(t)\} = \int_0^c e^{-st} f(t)\,dt + \int_0^\infty e^{-s(u+c)} f(u+c)\,du$$

$$= \int_0^c e^{-st} f(t)\,dt + e^{-sc} \int_0^\infty e^{-su} f(u)\,du \quad \text{(Since } f(u+c) = f(u) \text{)}$$

Periodic

$$\therefore L\{f(t)\} = \cancel{\int_0^c e^{-su} f(u)\,du + e^{-sc} \int_0^\infty e^{-su} f(u)\,du}$$

$$\cancel{= (1 + e^{-sc}) \int_0^c e^{-su} f(u)\,du}$$

$$= \int_0^c e^{-st} f(t)\,dt + e^{-cs} \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^c e^{-st} f(t)\,dt + e^{-cs} L\{f(t)\}$$

$$\Rightarrow (1 - e^{-cs}) L\{f(t)\} = \int_0^c e^{-st} f(t)\,dt$$

$$\therefore L\{f(t)\} = \frac{1}{1-e^{-cs}} \int_0^c e^{-st} f(t)\,dt.$$

Q. Find the L.T. of the square wave function of period $2c$, using the above example 1°.

Ex2° Find the Laplace transform of a square wave function

$$f(t) = \begin{cases} E, & \text{for } 0 \le t \le a/2 \\ -E, & \text{for } a/2 \le t < a \end{cases} \quad \text{and } f(t+a) = f(t).$$

Soln. Since $f(t)$ is a periodic function of period a, by Ex1° above

$$L\{f(t)\} = \frac{1}{1-e^{-as}} \int_0^a e^{-st} f(t)\, dt \quad —①$$

Now $$\int_0^a e^{-st} f(t)\, dt = \int_0^{a/2} e^{-st} f(t)\, dt + \int_{a/2}^{a} e^{-st} f(t)\, dt.$$

$$= \int_0^{a/2} e^{-st} E\, dt + \int_{a/2}^{a} e^{-st} (-E)\, dt.$$

$$= E\left[\frac{e^{-st}}{-s}\right]_0^{a/2} - E\left[\frac{e^{-st}}{-s}\right]_{a/2}^{a}$$

$$= E\left[\frac{1-e^{-sa/2}}{s}\right] - E\left[\frac{e^{-sa/2} - e^{-sa}}{s}\right]$$

$$= \frac{E}{s}\left[1 - 2e^{-sa/2} + e^{-sa}\right] = \frac{E}{s}\left(1-e^{-sa/2}\right)^2$$

From ①

$$L\{f(t)\} = \frac{E\left(1-e^{-sa/2}\right)^2}{s\left(1-e^{-sa}\right)} = \frac{E\left(1-e^{-sa/2}\right)^{\cancel{2}}}{s\left(1+e^{-sa/2}\right)\left(1-e^{-sa/2}\right)}$$

$$= \frac{E}{s}\left[\frac{1-e^{-sa/2}}{1+e^{-sa/2}}\right] = \frac{E}{s}\left[\frac{e^{sa/4} - e^{-sa/4}}{e^{sa/4} + e^{-sa/4}}\right] = \frac{E}{s}\tanh\frac{sa}{4}$$

Q. Find the Laplace transform of the function $f(t)$ with period $2\pi/\omega$ where

$$f(t) = \begin{cases} \sin \omega t, & \text{for } 0 < t < \pi/\omega \\ 0, & \text{for } \omega < t < 2\pi/\omega \end{cases}.$$

Q. Find the Laplace transform of the function

$$f(t) = \begin{cases} t, & \text{for } 0 < t < a \\ 2a-t, & \text{for } a < t < 2a \end{cases} \quad \text{and } f(t+2a) = f(t).$$

Soln. Since $f(t)$ is a periodic function of period $2a$

$$\therefore L\{f(t)\} = \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)\,dt \quad \text{— (1)}$$

Now $\int_0^{2a} e^{-st} f(t)\,dt = \int_0^a e^{-st}.\,t\,dt + \int_a^{2a} e^{-st}(2a-t)\,dt$

$$= \int_0^a \underset{II}{e^{-st}}.\underset{I}{t}\,dt + \int_a^{2a} e^{-st}.\,2a\,dt - \int_a^{2a} \underset{II}{e^{-st}}.\underset{I}{t}\,dt$$

$$= \left[-\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st}\right]_0^a + 2a\left[\frac{e^{-st}}{-s}\right]_a^{2a} - \left[-\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st}\right]_a^{2a}$$

$$= \left\{-\frac{a}{s}e^{-as} - \frac{1}{s^2}(1-e^{-as})\right\} + \frac{2a\,e^{-as}}{s}(1-e^{-as})$$

$$+ \left\{\frac{2a}{s}e^{-2as} + \frac{1}{s^2}e^{-2as} - \frac{a}{s}e^{-as} - \frac{1}{s^2}e^{-as}\right\}$$

$$= \frac{1}{s^2}\left(1-2e^{-as}+e^{-2as}\right) = \frac{(1-e^{-as})^2}{s^2}.$$

Q. Find the Laplace transform of $f(t) = \sin\left(\frac{\pi t}{a}\right)$ for $0<t<a$, the rectified wave function of period a.

Soln. $L\{f(t)\} = \int_0^a e^{-st}\sin\left(\frac{\pi t}{a}\right)dt$ = Imaginary part of $\int_0^a e^{-st} e^{i\frac{\pi t}{a}}\,dt$

$$= \text{Im. part}\left[\frac{e^{(-s+i\frac{\pi}{a})t}}{-s+i\pi/a}\right]_0^a = \text{Im part of}\left[\frac{e^{(-s+\frac{i\pi}{a})a}-1}{-s+i\pi/a}\right]$$

$$= \text{Im. part of}\left[\frac{e^{-as}(\cos\pi + i\sin\pi) - 1}{-s+i\pi/a}\right]$$

$$= \text{Im. part of} \left[\frac{1 - e^{-as}(\cos \pi + i \sin \pi)}{(s - i\pi/a)} \right]$$

$$= \text{Im part of} \left[\frac{(1 + e^{-as})\,(s + i\pi/a)}{(s - i\pi/a)(s + i\pi/a)} \right]$$

$$= \left[\frac{(1 + e^{-as})\frac{\pi}{a}}{(a^2s^2 + \pi^2)/a^2} \right] = \frac{(1 + e^{-as})\,a\pi}{a^2s^2 + \pi^2}$$

$$\therefore\ L\{f(t)\} = \frac{1}{1 - e^{-as}} \int_0^a e^{-st} f(t)\,dt.$$

$$= \frac{1}{1 - e^{-as}} \cdot \frac{(1 + e^{-as})\,a\pi}{a^2s^2 + \pi^2}$$

$$= \frac{e^{as/2} + e^{-as/2}}{e^{as/2} - e^{-as/2}} \cdot \frac{a\pi}{a^2s^2 + \pi^2}$$

$$= \frac{a\pi \coth \frac{as}{2}}{a^2s^2 + \pi^2}.$$